GLOBAL CONVERGENCE OF ITERATIVE PROCESSES

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We will deduce conditions under which the local convergence of iterative processes implies their global convergence.

1. Let us consider the equation

\[ T x = x, \quad (1.1) \]

where \( T \) is a nonlinear operator that maps a certain subset \( D \) of a (not necessarily complete) metric space \((X, d)\) into itself. We apply the usual iterative process for the solution of Eq. (1.1):

\[ x_{n+1} = T x_n \quad (n > 0), \quad (1.2) \]

where \( x_0 \in D \) is preassigned. Suppose that the solution set of Eq. (1.1) is nonempty. Let \( C(T) \) denote the domain of convergence of the iterative process (1.2):

\[ C(T) = \{x_0 \in D : d(x_0, F(T)) \to 0 \ (n \to \infty)\}. \quad (1.3) \]

We will say that process (1.2) converges globally if \( C(T) = D \).

Definition 1.1. Process (1.2) is said to be locally convergent if there exists a positive number \( c \) such that

\[ F_c = \{x \in D : d(x, F(T)) < c\} \subset C(T). \quad (1.4) \]

It is obvious that if process (1.2) is globally convergent, then it is locally convergent. Conversely, it has been proved in essence in [1] that if the operator \( T \) is stable on a convex set \( D \), then the local convergence of process (1.2) implies its global convergence. In what follows we will give some generalizations of this statement and also their applications.

Let us recall [1] that the operator \( T \) is said to be stable at a point \( x_0 \) if

\[ \forall \varepsilon > 0 \exists \delta = \delta(\varepsilon, x_0) \forall \varepsilon \in S(x_0, \delta) \cap D \forall n \geq 1 \ d(T^n x, T^n x_0) < \varepsilon. \quad (1.5) \]

Here, as also in the sequel, \( S(x_0, \delta) \) denotes the open ball with center at the point \( x_0 \) and radius \( \delta \). Operator \( T \) is said to be stable (on the set \( D \)) if it is stable at each point \( x_0 \in D \). In this connection, we introduce the notion of a quasistable operator.

Definition 1.2. Operator \( T \) is said to be quasistable at a point \( x_0 \in D \) if

\[ \forall \varepsilon > 0 \exists \delta = \delta(\varepsilon, x_0) \forall \varepsilon \in S(x_0, \delta) \cap D \forall n \geq 0 \ d(T^n x, T^n x_0) < \varepsilon. \quad (1.6) \]

If operator \( T \) is quasistable at each point \( x_0 \in D \), then it will be said to be quasistable (on the set \( D \)).

The discussion of the actual necessary and sufficient conditions for the stability and the quasistability of operators is beyond the scope of this note. Let us note only two facts: Firstly, each stable operator is quasistable and, secondly, the operator \( T \) in Example 2 given in [2] for other purposes is quasistable, but not stable, on the set

\[ \{(x, 0) : 0 \leq x \leq 1\} \subset \mathbb{R}^2. \]

2. At first, we formulate some simple propositions (see also [3, 4]).

**Lemma 2.1.** Suppose that process (1.2) is locally convergent for a continuous operator \( T \) that is defined on an open set \( D \). Then set \( C(T) \) is open.

**Proof.** This lemma is proved in the same manner as Theorem 4.1 of [4].

Conversely, the following lemma is obviously valid.
**Lemma 2.2.** If \( F(T) \) is a nonempty compact set and \( C(T) \) is an open set for a certain operator \( T \), then process (1.2) is locally convergent.

Let us observe that the openness of set \( C(T) \) means that process (1.2) is stable with respect to the initial approximation, i.e., if process (1.2) is convergent for a certain initial approximation \( x_0 \in D \), then it is also convergent for each initial approximation that is sufficiently near to \( x_0 \) and belongs to the set \( D \) of initial approximations. Thus, it follows from Lemmas 2.1 and 2.2 that if the operator \( T \) is continuous, the set \( D \) is open, and \( F(T) \) is compact, then the conditions for local convergence and for stability with respect to the initial approximation of the process (1.2) are equivalent.

**Theorem 2.3.** Suppose that a stable operator \( T \) maps the whole of a metric space \( (X, d) \) into itself. Then the local convergence of process (1.2) implies its global convergence.

**Proof.** Suppose that process (1.2) is locally convergent. Then, by Lemma 2.1, set \( C(T) \) is open. We prove that it is also closed. Then, by virtue of the connectedness of the space \( X \), we would get \( C(T) = X \), which would imply the global convergence of process (1.2).

Let us suppose that set \( C(T) \) is not closed. Then there exists a point \( x_0 \) that belongs to the closure of set \( C(T) \), but does not belong to the set itself. It follows from (1.3) that there exist numbers \( \varepsilon > 0 \) and \( n_k \) (\( k = 1, 2, \ldots \)) such that
\[
d(T^nx_0, F(T)) > \varepsilon
\]
for all \( n_k \) and \( n_k + 1 \) as \( k \to \infty \).

Since operator \( T \) is stable at the point \( x_0 \), by (1.5) there exists a positive number \( \delta = \delta(\varepsilon, x_0) > 0 \), such that \( d(T^n x, T^nx_0) < \varepsilon/2 \) for all \( x \in S(x_0, \delta) \). Since \( x_0 \) belongs to the closure of set \( C(T) \), there exists a point \( x_1 \in C(T) \) such that \( d(x_0, x_1) < \delta \). Further, by virtue of (1.3) there exists a positive number \( N \) such that \( d(T^nx_1, F(T)) < \varepsilon/2 \) for all \( n \geq N \).

Now if the number \( k \) is chosen such that \( n_k \geq N \), then we get
\[
d(T^{n_k}x_0, F(T)) < \varepsilon.
\]
This contradiction completes the proof of the theorem.

**Remark 2.1.** If set \( F(T) \) in the conditions of Theorem 2.3 is compact, then sequence (1.2) converges to a solution of Eq. (1.1) for each initial approximation \( x_0 \in X \). Indeed, since \( d(x_0, F(T)) \to 0 \), it follows by virtue of the compactness of \( F(T) \) that sequence \( \{x_n\} \) is also compact. Let \( p \) be a limit point of it. It is obvious that \( p \in F(T) \). We write down condition (1.5) for the stability of the operator \( T \) at the point \( p \):
\[
\forall \varepsilon > 0 \exists \delta = \delta(\varepsilon, x_0) > 0 \forall x \in S(p, \delta) \cap D
\]
\[
\forall n > 1 d(T^n x, p) = d(T^n x, T^n p) < \varepsilon.
\]
Since \( p \) is a limit point of the sequence \( \{x_n\} \subset D \) (here \( D = X \)), there exists a number \( n_0 = n_0(\varepsilon) \) such that \( x_n \in S(p, \delta) \cap D \). But then, by (2.1),
\[
d(x_n, p) = d(T^{n-n_0}x_n, p) < \varepsilon
\]
for all \( n > n_0 \). Thus, \( x_n \to p \in F(T) \).

**Theorem 2.4.** Suppose that a continuous quasistable operator \( T \) maps an arcwise connected set \( D \) into itself. Then the local convergence of process (1.2) implies its global convergence.

**Proof.** We take an arbitrary point \( x \in D \). Let us join \( x \) with a fixed point \( x_0 \in C(T) \subset D \) by a continuous path that lies entirely in \( D \), i.e.,
\[
x(t) \in D \quad (0 \leq t \leq 1), \quad x(0) = x_0, \quad x(1) = x.
\]
Let us set
\[
\alpha = \sup \{ t \in [0, 1]: x(t) \in C(T) \}. \quad (2.2)
\]
Since \( x_0 \in C(T) \), it follows that the number \( \alpha \) has been properly defined. Since process (1.2) is locally convergent, by (1.4) there exists a positive number \( \varepsilon \) such that \( F_\varepsilon \subset C(T) \). We write down condition (1.6) for the quasistability of the operator \( T \) at point \( x(\alpha) \in D \), for given \( \varepsilon \):
\[
\exists \delta = \delta(\varepsilon, x(\alpha)) \forall x \in S(x(\alpha), \delta) \cap D \forall N
\]
\[
\exists n = n(\varepsilon, x(\alpha), z) \geq N \forall d(T^nx, T^nx(\alpha)) < \varepsilon/2.
\]
We show that