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A PULLBACK THEOREM FOR COFIBRATIONS

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In this note we prove a pullback theorem for cofibrations, which extends a well known theorem of Strøm [5]. It also implies the pullback theorem of Heath [4] for locally equiconnected spaces. In addition, we comment on the dual problem of attaching fibrations.

I would like to thank the referee for his valuable comments.

1. THEOREM. Consider the commutative diagram in the category Top of topological spaces

$$\begin{array}{ccccc} X_o & \xrightarrow{f_o} & B_o & \xleftarrow{p_o} & E_o \\ \cap & & \cap & & \cap \\ X & \xrightarrow{f} & B & \xleftarrow{p} & E \end{array}$$

in which the inclusions are closed cofibrations. If p and p_o are fibrations then the inclusion $X_o x_{B_o} E_o \subset X\ x_B\ E$ is also a closed cofibration.

<u>Proof</u>: Since $B_o \subset B$ is a closed cofibration and p is a fibration, Strøm's well known pullback theorem ([5]th.12) implies that $p^{-1}(B_o) \subset E$ is a closed cofibration. Since $E_o \subset E$ is a closed cofibration, it then follows by [6]lemma 5 that $E_o \subset p^{-1}(B_o)$ is a closed cofibration. Now since p_o and $p_o'\colon p^{-1}(B_o) \rightarrow B_o$ (induced from p by $B_o \subset B$) are both fibrations, $E_o \subset p^{-1}(B_o)$ is a cofibration over B_o by [3], i.e. there exists a retraction r of $p^{-1}(B_o)$ x I onto $p^{-1}(B_o)$ x $\{0\} \cup E_o$x I such that $p\ pr_1 r(e,t) = p(e)$ for all $(e,t) \in p^{-1}(B_o)$ x I. It follows that the map

$$R\colon (X_o x_B\ p^{-1}(B_o))\ x\ I \rightarrow (X_o x_B\ p^{-1}(B_o))x\{0\}\cup(X_o x_{B_o} E_o)xI;$$
$$(x_o,e,t) \mapsto (x_o,r(e,t))$$

is well defined and a retraction too. Thus the inclusion $X_o x_{B_o} E_o \subset X_o x_B\ p^{-1}(B_o)$ is a (closed) cofibration.

On the other hand, $X_o \subset X$ being a closed cofibration and $\bar{p}: X \times_B E \to X: (x,e) \mapsto x$ being a fibration (pullback from p by f), again by [5] th. 12 it follows that $\bar{p}^{-1}(X_o) = X_o \times_B p^{-1}(B_o) \subset X \times_B E$ is a closed cofibration.

By composition, $X_o \times_{B_o} E_o \subset X \times_B E$ is a closed cofibration.

REMARKS. (i) The theorem generalizes a result of Furey and Heath ([2], prop. 1.7) and also implies the pullback theorem of Heath [4] for locally equiconnected spaces (using X LEC $\Leftrightarrow \Delta(X) \subset X \times X$ is a cofibration).

(ii) Strøm's pullback theorem 12 of [5] is the special case

$$\begin{array}{ccccc} B_o & \subset & B & \leftarrow & E \\ \cap & & \| & & \| \\ B & = & B & \leftarrow & E \end{array}$$

Also note that Strøm's theorem (and a fortiori our theorem) is no longer valid when "fibration" is replaced by "weak fibration". To see this let $p: E \to B$ be the weak fibration projecting the comb space (see [7], p.26) onto its base $B = I$ and consider the closed cofibration $\{0\} \subset I$. The reader will easily verify (using [1](2.29)) that $p^{-1}(0) \subset E$ is no cofibration (not even a weak cofibration).

COROLLARY. Consider the commutative diagram in Top

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & \cap & \searrow & \\ X & \xrightarrow{f} & B & \xleftarrow{p} & E \end{array}$$

in which the inclusions are closed cofibrations. If p is a fibration then the induced map $A \to X \times_B E$ is also a closed cofibration.

<u>Proof</u>: Just take $X_o = B_o = E_o = A$ and $p_o = f_o = 1_A$ in the theorem. Then $X_o \times_{B_o} E_o = \Delta(A)$ can be identified with A.

2. Dually, consider the commutative diagram in Top

$$\begin{array}{ccccc} Y' & \xleftarrow{f'} & A' & \subset & X' \\ \uparrow p_Y & & \uparrow p_A & & \uparrow p_X \\ Y & \xleftarrow[f]{} & A & \subset & X \end{array}$$