SULLIVAN'S LOCAL EULER CHARACTERISTIC THEOREM

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Using a certain cell decomposition of a closed neighborhood of a point $a$ in a real analytic set $A$ and the orientability modulo 2 of $A$ ([1,3.7] or [5,7.3]), we obtain a short proof, by counting cells, of D. Sullivan's theorem ([9]) that $\chi(A,A \sim \{a\})$ is odd.

1. INTRODUCTION. In [2] D. Burghelea and A. Verona gave a complete proof of this theorem using Alexander-Spanier cohomology, an analogue of Milnor's fibration theorem ([8, §4]), an interesting conic structure lemma for Whitney prestratifications, and Smith theory for the conjugation involution of $\mathbb{C}^n$. Our proof is based on a cell decomposition (Lemma 3) derived from a local Lojasiewicz stratification of $A$; the latter is established in [6, §11-15] or [7, §13] using the Weierstrass preparation theorem and classical elimination theory. The help of the referee in condensing the argument on page 5 is gratefully acknowledged.

2. DEFINITIONS. A subset $S$ of a real analytic manifold $M$ is called analytic (respectively, semianalytic) if $M$ admits a covering by open sets $U$ for which there is a real-valued function $f$ (respectively, a finite family $\mathcal{F}$ of real-valued functions) analytic in $U$ so that

$U \cap A$ equals $f^{-1}\{0\}$ (respectively, $U \cap A$ is a union of components of $f^{-1}\{0\} \sim g^{-1}\{0\}$ for some $f,g \in \mathcal{F}$). A locally-finite partition

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A of a subset \( I \) of \( M \) is called a semianalytic stratification of \( I \) if each \( S \in \mathcal{A} \) is a connected analytic submanifold such that \((I \cap \text{Clos } S) \sim S\) is a union of lower dimensional members of \( \mathcal{A} \).

Let \( H \) be any homology theory which treats the category of pairs of semianalytic sets and continuous maps between such pairs (for example, singular theory or real analytic theory [5]). For \( M \supset A \supset B \) semianalytic, we define

\[
I(A,B) = \sum_{j=1}^{\infty} (-1)^j \text{rank } H_j(A,B)
\]

whenever this sum has only finitely many nonzero terms. Let \( R^0 = \{0\} \), and, for any nonnegative integer \( m \) and \( x \in \mathbb{R}^m \), let \( p_0(x) = 0 \in R^0 \) and \( p_\ell(x) = (x_1, \ldots, x_\ell) \) whenever \( \ell \in \{1, 2, \ldots, m\} \) and \( x = (x_1, \ldots, x_m) \).

3. LEMMA. For any finite family \( \mathcal{A} \) of semianalytic subsets of \( \mathbb{R}^m \) and \( \varepsilon > 0 \), there exist an orthogonal transformation \( g \) of \( \mathbb{R}^m \), positive \( \delta_1, \ldots, \delta_m \) less than \( \varepsilon \), and, for integers \( 0 \leq k \leq \ell \leq m \), finite CW decompositions ([3,V,2.1]) \( g_\ell \) of

\[
I_\ell = \mathbb{R}^\ell \cap \{ (x_1, \ldots, x_\ell) : |x_1| \leq \delta_1, \ldots, |x_\ell| \leq \delta_\ell \}
\]

into semianalytic cells such that \( g_\ell = \{ p_\ell(D) : D \in \mathcal{A}_m \} \),

\[
\bigcup g_\ell \cap \{ D : \text{dim } D \leq k \} \text{ is analytic in } I_\ell, \text{ and, for each } S \in \mathcal{A},
\]

\( g(S) \cap \text{Int } I_m \) is a union of cells \( D \) in \( \mathcal{A}_m \) for which \( 0 \in \text{Clos } D \) and \( p_{\text{dim } S}|\text{Clos } D \) is a homeomorphism.

Proof: We use induction on \( m \). The case \( m = 0 \) being trivial, we assume \( m \geq 1 \) and abbreviate \( p = p_{m-1} \).