MULTIPLICITY OF POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENT IN SOME CONTRACTIBLE DOMAINS.

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In this paper we prove that, for every positive integer \( k \), there exists a contractible bounded domain \( \Omega \) in \( \mathbb{R}^N \) with \( N \geq 3 \), where the problem (*) (see Introduction) has at least \( k \) solutions.

Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with \( N \geq 3 \). In recent years there has been much interest in nonlinear elliptic equations of the form

\[
\begin{align*}
\Delta u + u^{2^* - 1} &= 0 \quad \text{in} \quad \Omega \\
\quad u > 0 \quad &\text{in} \quad \Omega \\
\quad u &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \( 2^* = \frac{2N}{N-2} \) is the critical exponent for the Sobolev imbedding \( H_0^{1,2}(\Omega) \subseteq L^p(\Omega) \).
The problem (\( \ast \)) is a simplified model of some variational problems in physics and geometry, whose common feature is a lack of compactness (see for example the Yamabe’s problem in [1], [25]).

Indeed, the solutions of (\( \ast \)) correspond to the critical points \( u \) of the functional

\[
\int_{\Omega} \left( \frac{1}{2} |Du|^2 - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} \right) dx, \quad \text{with } u > 0;
\]

but this functional does not satisfy the classical Palais-Smale’s condition, since the imbedding \( H^{1,2}_0(\Omega) \subset L^{2^*}(\Omega) \) is not compact; therefore it is not possible to use the standard variational methods to find critical points.

A first contribution to problem (\( \ast \)) is the following negative result due to Pohozaev.

**Theorem (Pohozaev [21]).** If the bounded domain \( \Omega \) is star-shaped, then (\( \ast \)) has no solution.

Nevertheless, more recently Brezis and Nirenberg have pointed out that lower-order perturbations of the nonlinear term in (\( \ast \)) can reverse this situation, and the perturbed problem can have solution, as follows also from general bifurcation theory (see [22], [19]).

Among the other results, Brezis and Nirenberg obtain in [6] the following theorem.

**Theorem (Brezis-Nirenberg [6]).** Let \( \Omega \subset \mathbb{R}^N \) with \( N \geq 3 \) and \( \lambda_1 \) denote the first eigenvalue of \(-\Delta\) in \( H^{1,2}_0(\Omega) \). There exists \( \lambda^* \) in \([0, \lambda_1]\), such that, if \( \lambda \in ]\lambda^*, \lambda_1[ \), then problem