A NOTE ON THE PROBLEM $-\Delta u = \lambda u + u|u|^{2^*-2}$

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The dual approach to the problem $-\Delta u = \lambda u + u|u|^{2^*-2}$, $u \in H^1_0(\Omega)$, permits a simple proof of a recent existence result [5] and allows extensions of this result to similar problems also with asymmetric nonlinearities.

1. This note deals with the Boundary Value Problem

(*) \[ -\Delta u = \lambda u + u|u|^{2^*-2} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \]

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$, $2^* = \frac{2N}{N-2}$ ($N > 2$) and $\lambda > 0$.

Extending a celebrated result of Brezis and Nirenberg [3], the following has been proved in [5]:

THEOREM. Let $N \geq 4$. For any $\lambda > 0$ (*) has a non-trivial solution.

Our purpose here is to present a new proof of this result (for $\lambda \notin \sigma(-\Delta)$), based on a dual variational ap-

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proach to (*), which allows a direct use of the Mountain-
Pass Theorem [1]. Such abstract setting permits to slight-
ly simplify the estimates and could allow further exten-
sions as indicated at the end of the paper.

2. Let \( \lambda_1, \lambda_2, \ldots \) be the eigenvalues of \(-\Delta\) on \(H^1_0(\Omega)\)
with corresponding eigenfunctions \(\phi_1, \phi_2, \ldots\), normalized by \(\int |\phi_k|^2 = 1\). Here and in the following \(\cdot = \int \cdot \cdot \cdot \). We write \(L^p = L^p(\Omega), H^1 = H^1_0(\Omega), \) etc. and denote by \(|\cdot|_p\) the norm in \(L^p\).

The solutions of (*) correspond to the critical points
of the functional \(J_\lambda\) on \(H^1_0\):

\[
J_\lambda(u) = \frac{1}{2} \int (|\nabla u|^2 - \lambda |u|^2) - \frac{1}{2*} \int |u|^{2*}.
\]

Let \(2^* = 2N/(N + 2)\) be the conjugate of \(2\). In the
following we will always take \(\lambda > 0, \lambda \neq \lambda_k, \ k = 1, 2, \ldots\). For
such \(\lambda\), let \(A_\lambda = (-\Delta - \lambda)^{-1} : L^{2^*} \subset H^{-1} \rightarrow H^1_0 \rightarrow L^{2^*}\), and con-
sider the functional \(J^*_\lambda\) on \(L^{2^*}\):

\[
J^*_\lambda(v) = \frac{1}{2^*} \int |v|^{2^*} - \frac{1}{2} \int v A_\lambda v.
\]

It is easy to verify that \(J^*_\lambda \in C^1\) with \(<dJ^*_\lambda(v), w> = \int wv|v|^{2^* - 2} - \int w A_\lambda v,\) and that \(v \in L^{2^*}\) is a critical point
for \(J^*_\lambda\) if \(u = v|v|^{2^* - 2} = A_\lambda v\) is a solution of (*).

Moreover the compactness properties of \(J_\lambda\) and \(J^*_\lambda\)
correspond. Precisely, let \(b \in \mathbb{R}\). We say that \(u_m \in L^{2^*}\)
(resp. \(H^1_0\)) is a \(PS_b\)-sequence for \(J^*_\lambda (J_\lambda)\) if: \(J^*_\lambda(u_m) \rightarrow b\) and \(dJ^*_\lambda(u_m) \rightarrow 0\) (resp. \(J_\lambda(u_m) \rightarrow b\) and \(dJ_\lambda(u_m) \rightarrow 0\)). \(J^*_\lambda (J_\lambda)\)
satisfies the \(PS_b\) condition if every \(PS_b\)-sequence has a
convergent subsequence. Concerning the \(PS\) condition, the
following holds [2, Lemma 5]:

**Lemma 1.** For all \(b \in \mathbb{R}\) \(J^*_\lambda\) satisfies \(PS_b\) iff \(J_\lambda\) does.

To keep the paper self-contained we sketch the proof.
Let \(J_\lambda\) satisfy \(PS_b\) and \(v_m\) be a \(PS_b\) sequence for \(J^*_\lambda\); then