A Pullback Theorem for Locally-Equiconnected Spaces

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ABSTRACT

In this note we prove a pullback Theorem for Locally Equiconnected spaces, that is dual to the well known adjunction Theorem of Dyer and Eilenberg [DE].

1. Statement of Results

Consider the following pullback in the category Top of Topological spaces.

\[
\begin{array}{ccc}
X \times E & \xrightarrow{f} & E \\
\downarrow{\bar{f}} & & \downarrow{p} \\
X & \xrightarrow{f} & B
\end{array}
\]

Thus \(X \times E\) is the subspace of \(X \times E\) consisting of pairs \((x,e)\) with \(f(x) = p(e)\). A space \(X\) is \textit{Locally Equiconnected} (LEC) if the inclusion of the diagonal \(\Delta X\) in \(X \times X\) is a cofibration. For Metric spaces this condition is easily seen to be equivalent to the classical definition, requiring that the diagonal be a strong neighbourhood deformation retract of \(X \times X\) (see [HN] for other conditions for this equivalence). The main result of this paper is the following:

\textbf{Theorem.} If \(p:E \to B\) is a fibration and \(E, B, \) and \(X\) are LEC then so also is \(X \times E\).

The theorem allows one to construct a Postnikov Tower that is LEC at each stage by using LEC spaces at each inductive step. This idea is dual to the result that CW complexes are LEC (see [L]). Our Theorem, though not the proof, is a kind of dual of the well known adjunction Theorem for locally
equiconnected spaces (see [DE] or [L]). Among other applications we have:

**Corollary A** The category of LEC spaces is under the formation of mapping tracks.

The next Corollary is stated but not proved in [H], it was announced to appear in the sequel to [HN], which unlikely to be completed.

**Corollary B** If \( p : E \to B \) is a fibration in which \( E \) and \( B \) are LEC then for any path \( \sigma : e \to e' \) in \( E \), there is a lifting function \( \lambda \) for \( p \) with the property that \( \lambda(\sigma, e) = \sigma \); Furthermore if \( p(\sigma) \) is not a constant path, then there is a regular lifting function with the same property as above.

If \( p : E \to B \) above has path connected fibres, then Corollary B allows one to choose translation functions between fibres that are base point preserving (see [H] p 282).

I would like to express my appreciation to the referee whose comments allowed for a shorter neater proof of the main result. The main question this work answers arose during joint work with G. Norton.

2. Proofs

The letter \( I \) will denote the closed unit interval \([0,1]\). We recall some definitions (see [tDKP]): Let \( A \subset X \) be a closed subspace of \( X \), a halo of \( A \) in \( X \), (called an N-halo in [HN]) is a map \( \psi : X \to I \) such that \( \psi^{-1}(0) = A \), \( A \) is said to be a strong halo deformation retract of \( X \) if there is a halo of \( A \) in \( X \) such that the subset \( V = \psi^{-1}([0,1]) \) is deformable to \( A \) in \( X \) rel. \( A \), that is there is a retraction \( r : V \to A \) and a homotopy \( H : V \times I \to X \) with \( H(x,0) = v \), \( H(x,1) = r(x) \), and \( H(a,t) = a \) for all \( x \in V \), \( t \in I \). The inclusion of a closed subspace \( A \) into a space \( X \) is a cofibration if \( A \) is a strong halo deformation retract of \( X \). Note that for us, all cofibrations are closed.

We name two results of Strøm that we will need: (i) The pullback rule [S2]. If \( p : E \to B \) is a fibration and \( A \to B \) is a cofibration, then the inclusion \( p^{-1}(A) \) into \( E \) is a cofibration. (ii) The power rule, [S3] If \( A \to X \) is a cofibration and \( C \) is compact, then the inclusion \( A^C \to X^C \) is also a cofibration, where \( X^Z \) denotes the space of continuous maps from \( Z \) to \( X \) with the compact open topology.