Computing the characteristic polynomial of a tree*

Bojan Mohar

Department of Mathematics, University of Ljubljana, Jadranska 19, 61111 Ljubljana, Yugoslavia

Received 3 November 1988
(received by the Publisher 20 September 1989)

Abstract

An algorithm is given for computing the values of the characteristic polynomial of a tree. Its time complexity is linear; hence, the polynomial is readily accessible from the tree and no computation is necessary to get the polynomial ready for applications. If necessary, the coefficients can be determined in time $O(n^2)$. This improves the complexity $O(n^3)$, reached by Tinhofer and Schreck, to $O(1)$.

1. Introduction

Tinhofer and Schreck [5] developed an algorithm with time complexity $O(n^3)$ for computing the characteristic polynomial of a tree. This is not a very surprising result, since the characteristic polynomial of any (symmetric) $n \times n$ matrix can be easily determined in time proportional to $n^3$ using the reduction to the tridiagonal form. In this note, we show that no computation is needed to determine the characteristic polynomial $\varphi(T;x)$ of a tree $T$. More precisely, for any value of $x$ we can determine the value $\varphi(T;x)$ in linear time, which means that the polynomial is readily accessible from the given tree. If we need the explicit coefficients of $\varphi(T;x)$, the application of our algorithm results in an $O(n^2)$ algorithm which calculates all the $n + 1$ coefficients of $\varphi(T;x)$.

We assume the basic knowledge of graph theory. Graphs are finite, undirected and simple. The characteristic polynomial $\varphi(T;x)$ of a graph $G$ is just the characteristic polynomial of the adjacency matrix of $G$ (cf. [2]).

*This work was supported in part by the Research Council of Slovenia, Yugoslavia.

© J.C. Baltzer AG, Scientific Publishing Company
2. Computing $\varphi(T; x)$

The following two results are well known (see, for example, [2,3]):

LEMMA 1

If $G_1, G_2, \ldots, G_k$ are components of the graph $G$, then

$$\varphi(G; x) = \varphi(G_1; x) \cdots \varphi(G_k; x).$$

LEMMA 2

Let $T$ be a forest and $v \in V(T)$. If $v_1, v_2, \ldots, v_d$ are the neighbours of $v$, then

$$\varphi(T; x) = x \varphi(T - v; x) - \sum_{i=1}^{d} \varphi(T - v - v_i; x).$$

Suppose that a tree $T$ is given and choose a vertex $v \in V(T)$. We shall assume henceforth that $T$ is a rooted tree with root $v$. (Otherwise, it can be transformed to such a form in linear time.) This means that each vertex $w \in V(T)$, except the root $v$, has a unique predecessor, the vertex $u \in V(T)$ which is the neighbour of $w$ and is closer to the root than $w$. The other neighbours of $w$ are its successors, and their number $\text{dout}(w) = \deg(w) - 1$ is called the out-degree of $w$. The root has $\text{dout}(v) = \deg(v)$.

If $w = v$, then let $T_w := T$. If $w \in V(T)$, $w \neq v$, has the predecessor $w'$, denote by $T_w$ the component of $T - \{ww'\}$ which contains $w$. Let $T'_w$ be the forest $T_w - w$. Notice that $T'_w = T_{w_1} \cup T_{w_2} \cup \ldots \cup T_{w_d}$, where $w_1, \ldots, w_d$ are the successors of $w$.

THEOREM 1

Let $w \in V(T)$.

(a) If $w$ is a leaf, i.e. $\text{dout}(w) = 0$, then

$$\varphi(T_w; x) = x \quad \text{and} \quad \varphi(T'_w; x) = 1.$$ 

(b) If $w_1, \ldots, w_d$ are the successors of $w$, then

$$\varphi(T'_w; x) = \prod_{i=1}^{d} \varphi(T_{w_i}; x)$$

and

$$\varphi(T_w; x) = \varphi(T'_w; x)(x - \sum_{i=1}^{d} \varphi(T'_{w_i}; x)/\varphi(T_{w_i}; x)).$$