ISOPARAMETRIC TRIPLE SYSTEMS OF FKM-TYPE II

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This paper continues the investigation of isoparametric triple systems of FKM-type started in [9]. We classify all such triple systems which are congruent to an isoparametric triple system of algebra type. We also consider the question to what extent the Clifford sphere of an FKM-triple is determined by the triple product. As an application the automorphism group of an FKM-triple is described.

For the geometric background of isoparametric triple systems the reader is referred to [6]§1 or [9] introduction. A short summary of the main algebraic properties of isoparametric triple systems is given below. The proofs of these results and all unexplained notions can be found in [6],[7] and [9]. Since this paper is a continuation of [9] we continue in the numbering of the sections and quote [9] without specific reference, e.g. Theorem 1.1 means [9] Theorem 1.1.

An isoparametric triple system is a tuple $(V,\langle\cdot,\cdot,\cdot\rangle,\{\cdot\})$ where $(V,\langle\cdot,\cdot,\cdot\rangle)$ is a finite-dimensional Euclidean vector space and $\{\cdot\}: V \times V \times V \longrightarrow V : (x,y,z) \longrightarrow \{xyz\} = T(x,y)z$ is a trilinear map (a so-called triple product) such that for all $x_1,x_2,x_3,x \in V$

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the following properties hold:

(ISO 1) \( \{x_1 x_2 x_3\} = \{x_\sigma(1) x_\sigma(2) x_\sigma(3)\} \) for any permutation \( \sigma \)

(ISO 2) the endomorphism \( T(x, y) \) is symmetric relative to \( \langle ., . \rangle \)

(ISO 3) \( \{xx\{xxx\}\} - 6\langle x, x\rangle\{xxx\} - 3\langle\{xxx\}, x\rangle x + 18\langle x, x\rangle^2 x = 0 \)

(ISO 4) there are positive integers \( m_1, m_2 \) such that

a) \( \text{trace } T(x, y) = 2(3 + 2m_1 + m_2) \langle x, y \rangle \) and

b) \( \dim V = 2(1 + m_1 + m_2) \).

If no confusion is possible, we write \( V \) or \( (V, \{\ldots\}) \) instead of \( (V, <.,.>, \{\ldots\}) \). We also use the abbreviation \( T(x) \) for \( T(x, x) \).

We remark that it is possible to give the defining identity (ISO 3) a shorter form by adding a scalar multiple of \( \langle x, x\rangle x \) to the old triple product: see Theorem 1.1. However, in the setting used here and in our papers [6]-[10] the connection between homogeneous isoparametric triple systems and non-reduced simple compact Jordan triple systems of rank 2 becomes more natural. This connection was very fruitful for the authors work on isoparametric triple systems in general.

Two isoparametric triple systems \( (V, \{\ldots\}_V) \) and \( (W, \{\ldots\}_W) \) are called isomorphic if there exists a bijective \( \mathbb{R} \)-linear map \( \phi : V \rightarrow W \) satisfying \( \phi\{xyz\}_V = \{\phi x\phi y\phi z\}_W \) for all \( x, y, z \in V \). An important feature of isoparametric triple systems is that they occur in pairs: If \( (V, \{\ldots\}) \) is isoparametric, then \( (V, \{\ldots\}') \), where \( \{xyz\}' = 3(\langle x, y\rangle z + \langle y, z\rangle x + \langle z, x\rangle y) - \{xyz\} \), is again isoparametric. We call \( (V, \{\ldots\}') \) the dual system of \( (V, \{\ldots\}) \) and abbreviate it by \( V' \) or \( (V, \{\ldots\}') \). Two isoparametric triple systems \( V \) and \( W \) are said to be equivalent (or congruent) if \( V \) is isomorphic to \( W \) or to \( W' \). It was shown in [6]§1 that there is a bijective correspondence between the equivalence classes of isoparametric hypersurfaces in spheres with 4 distinct principal curvatures and the equivalence classes of isoparametric triple systems.

The main tool in [6] and [7] was to consider tripotents