Second order effects in an elastic half-space acted upon by a non-uniform shear load

J. Guo and P. N. Kaloni, Windsor, Ontario

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Summary. A closed form solution to the second order elasticity problem, when an isotropic compressible elastic half-space undergoes a deformation owing to a non-uniformly distributed shear load, is presented. The method of integral transform is employed to determine the solutions. An example is discussed in detail to illustrate the second order effects. Numerical calculations for the second order elastic material for the z-direction displacement and the stress \( t_{zz} \) are carried out. It is found that the second order effect is to reduce the z-direction displacement and to decrease \( t_{zz} \) inside the circle but to increase its value outside the circle.

1 Introduction

In finite elasticity theory the mathematical equations governing the deformation of an isotropic compressible elastic material are highly nonlinear. As a result, the exact solutions of the boundary value problems have been possible in only some restricted cases, and often recourse has been taken to approximate methods. The method of successive approximations is one such technique which has received considerable attention. The second order solutions include terms which are quadratic in the displacement gradients, and obtaining a particular integral in explicit form becomes a formidable task. Rivlin [1] and Green and Spratt [2] were among the first to formulate the second order theories and a comprehensive account of the method is given by Truesdell and Noll [3] and Green and Adkins [4]. Goodman and Naghdi [5] have presented the use of displacement potentials for the solution of compressible second order elasticity problems and have applied it to plane strain problems. For incompressible materials a variety of techniques for the second order theories has been formulated by Chan and Carlson [6], Selvadurai and Spencer [7], Carroll and Mooney [8] and Lindsay [9]. Choi and Shield [10] have used the inverse deformation approach to study axisymmetric problems for a certain class of the second order elastic materials.

In the method of successive approximation, the displacements, stresses, etc. are expanded in a power series, in some suitable parameter, with non-zero radius of convergence. Signorini [11] and Stoppeli [12], [13] have discussed the results on existence and uniqueness of series solution under suitable differentiability conditions. Stoppeli, in particular, has shown that the displacement can be expanded as an absolutely convergent power series in some parameter, with non-zero radius of convergence, provided the parameter is sufficiently small and provided sufficiently smooth solutions of classical linear equations of elasticity exist.

In the present paper we follow Rivlin's [1] approach to treat the second order problem in a compressible elastic half-space which is acted upon by a non-uniformly distributed shear load. As will be noted below, Rivlin's approach reduces the second order problem to the solution of
a linear elasticity problem with body force. We recall that in the classical theory of elasticity the problem of stress distribution within an elastic half-space due to the action of a tangential force was first considered by Cerruti [15]. Besides the mathematical interest the solution to this problem was found to have practical usefulness in soil mechanics. Some generalizations to this and related problems of normal load when a non-uniform load is distributed over a circle were carried out by Sneddon [14]. Sneddon showed that such a solution of normal loading corresponds to the problem of determining the stress distribution when a half-space is deformed by the pressure against it of a rigid flat ended circular cylinder. We note that when the loading is distributed arbitrarily we need to consider both normal and shear loading, separately. We employ the method of integral transforms, as discussed by Sneddon [14] in both general linear and second order solutions. The general solutions are then specialized for a particular case and in the final Section, numerical calculations are carried out for displacement and stress in the z-direction.

2 Basic equations

If an elastic body undergoes deformation owing to a system of body forces \( X_i \) per unit mass of the material and surface forces \( X_{ni} \) per unit area of surface measured in the undeformed state of material, then the equations of equilibrium and boundary conditions are given as

\[
\frac{\partial \tau}{\partial H_{kj}} \frac{\partial}{\partial x_j} + q_0 X_i = 0 \tag{1}
\]

\[
X_{ni} = \frac{\partial \tau}{\partial H_{ks}} l_{ik} \tag{2}
\]

where \( \tau_{ik} \) are the Cauchy stresses and \( u_i \) are the displacements, \( x_k \) are the coordinates of the undeformed particle of the elastic body and \( l_i \) are the direction-cosines of the normal to the undeformed surface.

Also

\[
H_{ik} = \frac{\partial u_i}{\partial x_k}, \quad \tau = \det \left( \delta_{ik} + \frac{\partial u_i}{\partial x_k} \right), \quad (i, k = 1, 2, 3).
\]

By considering the strain energy function \( W \), for a compressible isotropic material, to be of the form

\[
W = a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_3^3 + a_5 J_3 \tag{3}
\]

where \( a_1 \) to \( a_5 \) are material constants and \( J_1, J_2 \) and \( J_3 \) are strain invariants of the first, second and third order in strains respectively, Rivlin [1] has shown that Eqs. (1) and (2), neglecting terms of higher degree than the second on the space derivative of displacement \( u_i \), can be written, respectively, as

\[
\left[ (1 + \alpha) \frac{\partial u_k}{\partial x_k} \right] \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0 \tag{4}
\]