The geometry of uniformity in second-grade elasticity

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Summary. The underlying structure of the theory of continuous distributions of defects, such as dislocations and disclinations, in second-grade elastic bodies is presented in terms of second-order G-structures.

1 Introduction

Since the early work of Kondo [10], Bilby [7], Kröner [9], and others, it was understood that the theory of continuous distributions of defects in crystalline media finds its natural expression in the language of modern differential geometry. Later work by Noll [5] and Wang [6] on the theory of inhomogeneities in continuous media, while making no mention, heuristic or otherwise, of the underlying crystalline structure, led to similar conclusions. More recently, the realization [3] that the basic geometric object involved in the formulation is a G-structure, has permitted the derivation of specific results. In extending this last approach to second-grade elastic materials, we bear in mind that the precise connection between some physical quantities, such as disclinations and couple-stresses, and their putative geometric counterparts, remains still an open question, and that it is within the realm of second-grade elasticity that the natural answer is likely to be found. The main aim of this paper, however, is to introduce the basic idea of second-grade homogeneity as the physical counterpart of the integrability of a certain, precisely defined geometric object: a second-order G-structure. This object, although well established, is still in need of further mathematical study. We have, nevertheless, refrained from presenting new mathematical results, some of which have appeared elsewhere [2], and we have, rather, focussed on the most elementary presentation possible, given the complexity of the subject. In the final Section, an attempt to foster an intuitive picture of the rather complicated situation at hand is presented. We hope that the interest of mechanicists in this fertile area of research will be aroused or renewed.

2 Review of the first-grade case

A material body \( B \) is a three-dimensional differentiable manifold that can be covered with just one chart

\[ \Phi : B \to \mathbb{R}^3, \]

which may be interpreted physically as a configuration in a Euclidian space referred to a fixed basis. By composition, it is possible (and convenient) to refer all configurations \( \Phi \) to a fixed
arbitrarily chosen reference configuration $\Phi_0$ by means of the deformation map

$$\chi = \Phi \circ \Phi_0^{-1},$$

where the inverse map is understood to act over the image $\Phi_0(B)$ of $B$.

A body $B$ is said to be made of a first-grade elastic material if its mechanical response at each point $X$ of $B$ is completely characterized by one or more functions $W$ of the value of the deformation gradient

$$F(X) = \nabla \chi(X)$$

at that point, namely,

$$W = W_0(F; X).$$

The dependence of $W_0$ on the material point $X$ may be genuinely due to the fact that different points of the body are made of different materials, in which case the body is non-uniform. But it may also happen that the body is uniform and that $W_0$ nevertheless depends on $X$ because, perhaps, of an “unhappy” choice of reference configuration. Indeed, if $\lambda$ is a deformation representing a change of reference configuration from, say, $\Phi_0$ to $\Phi_1$, then the material response $W_1$ with respect to $\Phi_1$ is given by

$$W = W_1(F; \lambda(X)) = W_0(F \circ \nabla \lambda; X).$$

If the body is uniform there exists at each point $X$ a non-singular linear map $P(X)$

$$P(X) : \mathbb{R}^3 \rightarrow T_\chi \Phi_0(B),$$

mapping a reference crystal in $\mathbb{R}^3$ onto tangent vectors at the point $X$ of the reference configuration, such that

$$W(F; X) = \tilde{W}(FP(X)), \quad (1)$$

identically for all non-singular $F$, where $\tilde{W}$ is a function of one matrix variable only.

It should be noted that the uniformity field $P(X)$ is not necessarily unique, nor is its inverse necessarily the gradient of a global deformation. If there exists at least one field $P$ satisfying the uniformity condition (1), and whose inverse is the gradient of a deformation, the body is said to be globally homogeneous. If for each point $X$ of $B$ a deformation can be found such that the inverse of the $P$-field is equal to its gradient in a neighborhood of $X$, the body is said to be locally homogeneous. If the uniform body fails to be locally homogeneous, it is called inhomogeneous. Physically, in this case no change of reference configuration can remove the dependence of the response on the material point, even locally.

As noted above, for a given uniform body, the field of uniformity maps $P(X)$ need not be unique. In fact, a multiplication to the right by a constant tensor $A$ trivially yields the new uniformity field $P(X)A$. This mathematical operation can be easily identified with the physical one of simply changing the reference crystal. But, in addition to this trivial non-uniqueness, there is an essential, point-dependent, one stemming from the possible existence of material symmetries. If $G$ is the symmetry group of the chosen reference crystal, and $P(X)$ is a given uniformity field, then the totality of possible uniformity fields is the set

$$\mathcal{P}(X) = P(X) G. \quad (2)$$

Attaching at each point $X \in \Phi_0(B)$ the corresponding set $\mathcal{P}(X)$, Fig. 1, it can be shown [2] that the resulting object can be viewed as a $G$-structure with $G$ as its structural group.