Integral Representations in Hydrodynamics

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(Received January 10, 1983)

Summary

In this paper, general integral representations are obtained for the unknown velocity field of the Stokes equations for both bounded and unbounded domains. These representations lead to some interesting results including the well-known uniqueness property of the so-called exterior Stokes problem.

1. Introduction

In a previous work [1], the Stokes equations were examined and an integral representation was obtained for the unknown velocity field in the case of three-dimensional flow. This representation was however, restricted to only points in the interior of a bounded domain. The main objective of this note is to extend this result and to consider some applications.

Fundamental singular solutions are generated for the three-dimensional Stokes equations using a Helmholtz decomposition and a Fourier transform. With the aid of these solutions and a reciprocal theorem, a general integral representation with respect to a bounded domain is obtained. The case of an infinite domain is also considered and a similar integral representation is obtained assuming certain supplementary asymptotic conditions. These representations lead to some important results including the uniqueness of the exterior Stokes problem.

2. Fundamental Solutions

The equations of motion characterizing the slow steady flow of a Newtonian fluid are given by

\[ t_{ij,j} + f_i = 0 \]
\[ t_{ij} = t_{ij}, \] (1)

while the constitutive law takes the form

\[ t_{ij}(u) = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \] (2)

Here \( u, p, f, t_{ij}, \mu \) and \( \delta_{ij} \) represent respectively the velocity vector, the pressure, the body force, the components of the Cauchy stress tensor, a material coefficient and the Kronecker delta. In the case of incompressible flow, the above equations
(1) and (2) generate the so-called Stokes equations
\[ \mu \nabla^2 u - V p = -f, \]
\[ V \cdot u = 0. \]

Let us introduce the following Helmholtz decomposition:
\[ u = \nabla \times \Phi; \quad V \cdot \Phi = 0, \]
\[ f = \nabla \phi + \nabla \times \Psi; \quad V \cdot \Psi = 0. \]

Substituting (4) and (5) into (3) gives
\[ \nabla^2 \Phi + \frac{1}{\mu} \Psi = 0, \]
\[ P = \phi. \]

If \( \widetilde{g}(s) \) denotes the Fourier transform of \( g(x) \), then, by definition
\[ \widetilde{g}(s) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{-is \cdot x} dx. \]

Taking the Fourier transform of (6), we get
\[ \widetilde{\Phi}(s) = \frac{1}{s^2 \mu} \widetilde{\Psi}(s). \]

This is the transformed solution and the inverse transform can be taken if the body force is known. The case of a concentrated point force in an infinite unbounded medium otherwise at rest will now be examined. Let this force be represented by
\[ f = F \delta(x - y), \]
where \( F \) is a constant vector and \( \delta(x - y) \) is the Dirac delta function.

The scalar and vector potentials satisfying (5) may be represented by
\[ \phi(x) = -\frac{1}{4\pi} F \cdot V \left( \frac{1}{r} \right), \]
\[ \Psi(x) = -\frac{1}{4\pi} F \times V \left( \frac{1}{r} \right) \]
where \( r^2 = \sum_{i=1}^{3} (x_i - y_i)^2 \). Taking the Fourier transform of (10) and substituting into (8) gives
\[ \widetilde{\Phi}(s) = -\frac{F \times is}{\mu s^2}. \]

On taking the inverse transform of (11) and substituting into (4), we obtain
\[ u = \frac{1}{8\pi \mu} \left[ \frac{F}{r} + \frac{(F \cdot r)}{r^2} \right], \]