The Circular Leaf Spring

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With 7 Figures

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Summary — Zusammenfassung

The Circular Leaf Spring. Large deflections of the simple circular leaf spring are analyzed using elastica theory. The governing equations are integrated by a new homotopy method which gives accurate results. The force-displacement and maximum moment-displacement curves are obtained for various natural curvatures. It is found that these curves are highly nonlinear. Springs with large initial curvature exhibit nonuniqueness. For the same sufficiently large force the spring equilibrates theoretically in three different configurations, one of which is unstable. Only one of the two stable configurations is applicable to the leaf spring.


Nomenclature

\(a\) initial guess vector
\(b\) vertical position of spring tip
\(EI\) equivalent flexural rigidity
\(F\) terminal force
\(L\) half arc length of spring
\(M\) maximum moment
\(M_0\) natural curvature
\(\varphi\) homotopy map
\(\ell\) dimensional quantities
\(R\) natural radius
\(s\) arc length of spring
\(t\) arc length of zero curve of \(\varphi\)
\(v\) solution vector
\(w\) vector function
\(x\) cartesian coordinate
\(y\) cartesian coordinate
\(\theta\) local angle of inclination
\(\lambda\) parameter
Introduction

The leaf spring is an extremely important industrial suspension mechanism. Figure 1 shows a simple leaf spring composed of two identical precurved circular flat springs hinged at the ends. If the thickness of the spring is small compared to the length, the spring may be considered as an elastica. The theory of elastica was first formulated by Euler [1] who stated that the curvature of a thin rod at any point is proportional to the local moment applied. One may refer to a noteworthy book by Frisch-Fay [2] for the development of elastica theory prior to 1962.

The only theoretical work on the simple leaf spring is due to Frisch-Fay [3] in 1960. He solved the elastica equations using elliptic functions. The result is applied to a spring of specific dimensions and rigidity. The present work differs from that of Frisch-Fay in the following aspects.

Firstly, the use of elliptic functions is extremely inconvenient. The important design parameters, force and deflection, are implicit in the solution involving elliptic functions. One can only obtain these parameters *aposteriori* from an assumed terminal angle and an elastic modulus. In other words, force and deflection cannot be controlled. One then thinks of using Newton-Raphson iteration to search for the correct terminal angle and elliptic modulus, given a force or deflection. As noted by Shoup [4, 5], this scheme fails to converge unless the initial guess is extremely close to the correct solution. In this paper we shall use a new homotopy method which, for a given force or deflection, requires neither elliptic functions nor a close initial guess for convergence.

Secondly, Frisch-Fay [3] considered only one specific leaf spring, i.e., one whose arc length to the natural radius of curvature was 26/15. We shall compute the force-displacement and maximum moment-displacement curves for arbitrary natural curvatures. These curves are very important in the design of leaf springs.

Formulation

Fig. 1 shows the leaf spring can be considered as an initially curved cantilever subjected to a terminal load. The governing equations are

\[ EI \frac{d\theta}{ds'} = M_0' - M' + F'x' \]  \hspace{1cm} (1)

\[ \frac{dx'}{ds'} = \cos \theta, \quad \frac{dy'}{ds'} = \sin \theta. \]  \hspace{1cm} (2)

![Fig. 1. The leaf spring and the coordinate axes](image_url)