manuscripta math. 38, 325 - 332 (1982)

manuscripta mathematica

THE "WORLD'S SIMPLEST AXIOM OF CHOICE" FAILS

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We use topos-theoretic methods to show that intuitionistic set theory with countable or dependent choice does not imply that every family, all of whose elements are doubletons and which has at most one element, has a choice function.

1. Introduction

The axiom in question (WSAC), originally formulated by F. Richman, states, $\forall F$

$$\begin{array}{l} \forall x,y \in F \cdot x = y \\ \wedge \\ \forall x \in F \cdot \exists v,w \cdot (v \neq w \quad x = \{v,w\}) \end{array} \rightarrow \exists f : F \rightarrow \cup F \ .$$

Of course, for such a family, any such f is a choice function, $\forall x \in F \cdot f(x) \in x$. Also, F is completely determined by $\cup F$

$$F = \{\cup F \mid \exists v \cdot v \in \cup F\} \ .$$

Furthermore, $\cup F$ may be any set which if it is inhabited is a doubleton. If A is such that

$$\exists v \cdot v \in A \rightarrow \exists v,w \cdot (v \neq w \quad A = \{v,w\}),$$

let $F = \{A \mid \exists x \cdot x \in A\}$, then $A = \cup F$.

We use the technique (originating with Joyal) of considering the universal example of such a set A. It is easy to check that the interpretation of higher-order logic in the classifying topos [12] [16] satisfies countable and dependent choice, CC & DC but not WSAC. This is perhaps the world's simplest example of this technique. We show that such examples can arise in the well-founded part of a topos [7] by embedding this universal example

0025-2611/82/0038/0325/$01.60

universally in 2^N (as in [3] §5).

For details on classifying topoi we refer to Tierney [16] or Makkai & Reyes [11]. The interpretation of intuitionistic type-theory and set-theory in a topos is described by Fourman [2] and [3] and more concretely, by Osius [13] and Scott [15].

2. The Basic Model

We introduce universally a set A which if it is inhabited is a doubleton. The classifying topos is the functor category $S^{\mathbb{C}}$ where $\mathbb{C}$ is the category whose non-identities are

$$E \xrightarrow{\alpha} D \circlearrowleft \beta$$

with $\beta \circ \alpha = \alpha$ and $\beta^2 = \mathrm{id}$. This is (equivalent to) the category of finitely presented such sets (E is the empty set and D a doubleton) and monomorphisms (monos because the equality on such a set is decidable and this would be reflected in any geometric axiomatisation by the addition of predicates or operations forcing homomorphisms to be monos).

The models we consider in this section are thus functors from $\mathbb{C}$ into the category of sets. These are like Kripke models. The category $\mathbb{C}$ replaces the usual partial order and instead of restrictions we need transition maps corresponding to the morphisms of our category. In particular, in our case, a model X will be a pair of domains $X(E)$ and $X(D)$ together with an automorphism $X(\beta) : X(D) \to X(D)$ of order two and a restriction map $X(\alpha) : X(E) \to X(D)$ whose image is fixed by $X(\beta)$. The interpretation of logic in such a presheaf topos is given by a straightforward generalisation of Kripke's definitions [9]. Function spaces are modelled by the categorical exponents which are easily calculated using the Yoneda lemma [11] [10].

The universal A we want is given by the forgetful functor $\mathbb{C} \to \mathrm{Sets}$. The family $F = \{A \mid \exists x,\ x \in A\}$ is