THE INJECTIVE HULL OF AN OPERATOR IDEAL ON
LOCALLY CONVEX SPACES

L. Franco and C. Piñeiro

In his book [3], Pietsch presents the problem of a direct
construction of the injective hull of an operator ideal on loca-
ly convex spaces. We construct the injective hull of an arbi-
trary operator ideal on locally convex spaces using the functor
$E \in \mathcal{L} \mapsto E_{\mathcal{B}} \in \mathcal{L}$, where $E_{\mathcal{B}} = \bigcap \mathcal{E}(B, K)$ and $B$ cover the family
of all equicontinuous subsets of $E'$. If $E'$ is the category of all
locally convex spaces. If the operator ideal is bounded, we ob-
tain its injective hull using seminorm ideals.

1. Introduction

At first some concepts of the theory of operator ideals
in the sense of Pietsch [3] which are necessary for the follo-
wing description shall be given.

A subclass $\mathcal{U}$ of the class $\mathcal{L}$ of all continuous operators
between locally convex spaces (l.c.s.) will be called an operator
ideal, if the components $\mathcal{U}(E, F) = \mathcal{U} \cap \mathcal{L}(E, F)$ satisfy the fo-
llowing conditions:

(I$_1$) $\mathcal{U}(E, F)$ contains the class $\mathcal{F}(E, F)$ of all finite di-
mensional operators.

(I$_2$) $\mathcal{U}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$.

(I$_3$) If $T \in \mathcal{L}(E, F)$, $S \in \mathcal{U}(E, F)$ and $R \in \mathcal{L}(F, F)'$, then

$R \cdot S \cdot T \in \mathcal{U}(E, F)'$.

By bounded operators we mean those which carry some neigh-
orhoods to a bounded set. A subclass $\mathcal{U}$ of the class $\mathcal{A}$ of all
bounded operators between l.c.s. will be called a bounded opera-
tor ideal, if the components $\mathcal{U}(E, F) = \mathcal{U} \cap \mathcal{A}(E, F)$ satisfy the
previous conditions.
The ideal $\mathcal{U}$ will be called injective, if any operator $T \in \mathcal{L}(E,F)$ belongs to $\mathcal{U}(E,F)$, as soon as there exists an isomorphic embedding $J : F \rightarrow F_0$ such that $J \cdot T \in \mathcal{U}(E,F_0)$.

Since the intersection of injective operator ideals is also injective, for every given operator ideal $\mathcal{U}$ we define an injective hull which will be designed $\mathcal{U}^i$.

For each l.c.s. $E$, we denote by $\mathcal{G}(E)$ the set of all continuous seminorms on $E$.

A subclass $\mathcal{G}'$ of the class $\mathcal{G}$ of all continuous seminorms on l.c.s. will be called a seminorm ideal, if the components $\mathcal{G}'(E) = \mathcal{G} \cap \mathcal{G}(E)$ satisfy the following conditions:

1. For each $a \in E'$ there exist $p \in \mathcal{G}'(E)$ and $\alpha > 0$ such that $|\langle x, a \rangle| < \alpha \cdot p(x)$ for every $x \in E$.
2. If $p_1, p_2 \in \mathcal{G}'(E)$ then there exists $p \in \mathcal{G}(E)$ such that $p < p_i$ ($i = 1, 2$).
3. For every $q \in \mathcal{G}(F)$ and $T \in \mathcal{L}(E,F)$ there exist $p \in \mathcal{G}(E)$ and $\alpha > 0$ such that $q(Tx) \leq \alpha \cdot p(x)$ for every $x \in E$.
4. If $p \in \mathcal{G}(E)$ and $q < p$, then $q \in \mathcal{G}(E)$.

It is easy to verify that the conditions $(S_1) \rightarrow (S_4)$ are equivalent to the following assertion: $\mathcal{G}'(E)$ defines a topology $\tau_{\mathcal{G}'(E)}$ in $E$ that verifies:

1. $\tau_{\mathcal{G}'(E)}$ is a l.c. topology finer than $\tau_{(E,E')}$ and coarser than the topology $\tau$ of $E$.
2. If $T \in \mathcal{L}(E,F)$ then $T \in \mathcal{L}((E,\tau_{\mathcal{G}'(E)}),(F,\tau_{\mathcal{G}(F)}))$.

In the theory of Banach spaces, Pietsch [4] obtains constructions of the injective hull of an operator ideal by two well-different methods, using in one case seminorm ideals and in the other case joining to each Banach space $E$, the space $E_\infty$ of the bounded functions in the polar of unit ball. In [3], Pietsch presents the problem of a direct construction of the injective hull of an operator ideal on l.c.s. In section 2 we construct the injective hull of a bounded operator ideal on l.c.s. using seminorm ideals. In section 3 we study the properties of functor $E \in L \mapsto E_\infty \in L$, where $L$ denotes the category of all l.c.s. and $E_\infty = \prod \mathcal{G}(E,K)$,