ON QUASIBARRELLED SPACES (*)

A. Marquina and P. Pérez Carreras

In this paper we introduce two new classes of quasibarrelled spaces the strong barrelled spaces and the semibornological spaces. In order to show that the semibornological are a strict subclass of the quasibarrelled spaces, we prove the completeness of the strong dual. We also show that every quasibarrelled (DF)-space is semibornological. The hereditary properties are also studied and it is shown that the product of a non countable family of non-zero dimension strong barrelled (semi-bornological) space is a strong barrelled (semi-bornological) spaces, using a method due to M. Valdivia. The same method is worth to find classes of strong barrelled spaces which are not bornological. It remains as open question if every finite codimensional subspace of a semi-bornological (strong barrelled) space is semibornological (strong barrelled).

The linear spaces we use here are defined over the field $k$ of the real or complex numbers. We denote by $E[T]$ the linear space $E$ provided with the separated locally convex topology $T$. As it usual, we write $E'$ and $E^\mathfrak{a}$ for the topological and algebraical dual space of $E$, respectively. Given a bounded closed absolutely convex set $B$ in $E$, we denote by $E_B$ the normed space over the linear hull of $B$, with $B$ as closed unit ball. We use the symbols $\sigma(E,F)$, $\nu(E,F)$, $\beta(B,F)$ to denote the weak, Mackey and strong topologies, respectively, being $\langle E,F \rangle$ a dual pair.

The following definition is due to Valdivia:

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DEFINITION 1. Given a space $E[T]$, let $A$ be a family of closed absolutely convex sets of $E$, which cover $E$, and such that $A$ is invariant by scalar multiplication and let $B$ be a family of sets of $E$ such that:

1) If $B \subseteq B$, then $B$ is a closed bounded absolutely convex set of $E$. II) If $B_1, B_2 \in B$, there exists a $B_3 \in B$ such that $B_1 \cup B_2 \subseteq B_3$. III) If $B \subseteq B$, then $\lambda B \subseteq B$ for every $\lambda \in K$, $\lambda \neq 0$ and so that $B$ covers $E$. We say that $E[T]$ is $AB$-barrelled if, and only if, every absolutely convex set $U$ of $E$ which absorbs the elements of the family $B$ and intersects the elements of the family $A$ in closed sets of $E$, is $T$-neighbourhood of the origin in $E$. We call $AB$-barrel to a set $U$ of that kind.

From the definition it is easy to see that a space $E[T]$ is barrelled if, and only if, $E[T]$ is $AB$-barrelled being $B$ the family of the absolutely convex hulls of the finite sets of $E$ and $A$ constituted by $E$, as unique element. If we take $B$ for the family of all the bounded closed absolutely convex sets of $E$, we get the quasibarrelled spaces.

It is easy to show that the algebraic closure of an absolutely convex set in a space $E[T]$, is its topological closure for the finest locally convex topology on $E$. Using this fact, every absolutely convex set of a space $E[T]$ contains the algebraic closure of $\frac{1}{2} U$. Indeed, $U$ is neighbourhood of the origin in its linear hull, provided with the finest locally convex topology, and so $\frac{1}{2} U \subseteq U$ and $\frac{1}{2} U$ coincides with the algebraic closure of $\frac{1}{2} U$. Thus, a space $E[T]$ is bornological if, and only if, every bornivorous algebraic closed absolutely convex set of $E$ is $T$-neighbourhood of the origin. We know that a absolutely convex set is algebraic closed if, and only if, it intersects the subspaces of finite dimension in algebraic closed sets of $E$; then a space $E[T]$ is bornological if, and only if, $E$ is $AB$-barrelled being $A$ the family of all the bounded closed absolutely convex sets of $E$, which generate finite dimensional subspaces of $E$, and $B$ the family of all bounded closed absolutely convex sets of $E$.

Our next goal is characterize the ultrabornological spaces as $AB$-barrelled spaces. We say that a set $H$ of a space $E[T]$ is $B$-compact if there exists an absolutely convex closed and bounded set $L$ of $E$, such that $H$ is compact in $E_L$ and $E_L$ is a Banach space. The following is a well known result: a) In a complete bornological space, every absolutely convex set of $E$, which absorbs the compact absolutely convex