Analytical study of the thermal shock problem of a half-space with various thermoelastic models

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Summary. The thermal shock problem of a half-space is frequently used as a test example for checking computer programs which are suitable for solving thermoelastic problems. As there are no exact solutions for the problem formulated on the basis of complicated models, authors have compared their own results to previous numerical results in their publications so far. In the present paper six thermoelastic models are investigated. Curves of temperature, displacement and stress fields arising as a result of sudden and ramp-type surface heating are given in a very handy form for program testing. A thorough analysis of elastic and thermal waves is also included.

1 Introduction

In case of homogeneous, isotropic continuum, the governing equations of the classical linear theory of thermoelasticity for displacement and temperature fields consist of the following coupled differential equations [1], [2]

\[
\frac{E}{2(1 + v)} \left( u_{i,kk} + \frac{1}{1 - 2v} u_{k,ki} \right) + \rho f_i - \rho \ddot{u}_i - \frac{E \alpha}{1 - 2v} T_{,i} = 0, 
\]

\[
\lambda T_{,kk} + r - \rho c_v \dot{T} - \frac{E \alpha T_0}{1 - 2v} \dot{u}_{k,k} = 0,
\]

in which \( E \) is Young’s modulus; \( v \) is Poisson’s ratio; \( u_i \) and \( f_i \) are the Cartesian components of displacement and body force vector, respectively; \( \rho \) is the mass density; \( \alpha \) is the linear thermal expansion; \( T \) is the absolute temperature; \( \lambda \) is the thermal conductivity; \( r \) is the heat source; \( c_v \) is the specific heat at constant strain and \( T_0 \) is the reference temperature of the natural, stress-free state. Superposed dots (\( \cdot \)) and commas (\( ,_i \)) are used to denote material time differentiation and partial differentiation with respect to Cartesian co-ordinates \( x_i (i = 1, 2, 3) \), respectively.

The equation of motion (1) is of wave-type, a hyperbolic equation, while the classical heat-conduction equation (2) is of diffusion-type, a parabolic equation. The latter equation implies that the thermal disturbances propagate with infinite velocity. Since the equations are coupled, the propagation of mechanical disturbances has similar characteristics. To eliminate this physically unrealistic behaviour, generalized theories of thermoelasticity have been proposed (e.g. Lord and Shulman [3], Green and Lindsay [4]) to allow for so-called “second sound” effects. It is the generalized theory developed by Lord and Shulman
in 1967 that will be used in this paper. This theory includes the following generalized heat-conduction equation

\[ \lambda T_{,kk} + (r + \tau_r \dot{T}) - \varrho c_v (\dot{T} + \tau_r \ddot{T}) - \frac{E \alpha T}{1 - 2\nu} (\ddot{u}_{k,k} + \tau_r \dddot{u}_{k,k}) = 0, \] (3)

where \( \tau_r \) is the thermal relaxation time.

In the engineering practice, adequate results can be obtained for many problems using various simplified models instead of the above fully coupled equations of thermoelasticity. The simplification usually means that the inertia term in the equation of motion (1) is neglected, or the coupling terms are eliminated from the heat-conduction equations (2) or (3), that is

\[ \frac{E}{2(1 + \nu)} \left( u_{i,kk} + \frac{1}{1 - 2\nu} u_{k,ki} \right) + \rho \ddot{u}_i - \frac{E \alpha}{1 - 2\nu} T_{,i} = 0, \] (4)

\[ \lambda T_{,kk} + r - \varrho c_v \dddot{T} = 0, \] (5)

\[ \lambda T_{,kk} + (r + \tau_r \dot{T}) - \varrho c_v (\dot{T} + \tau_r \ddot{T}) = 0. \] (6)

The thermoelastic models most frequently occurring in technical literature are summarized in Table 1. Further on, these models will be referred to also by their serial numbers given in the Table.

The class of thermoelastic problems which can be solved analytically is very small. Therefore, much effort has been made during the last two decades to develop numerical methods for solving thermoelastic problems. The developed (mainly finite element) programs are, however, reliable in practice only if they have been tested very thoroughly. Thus, in respect of numerical methods, it is very important to determine some reference solutions. Because of its relative simplicity, the thermal shock problem of a half-space seems to be suitable for getting such solutions.

Analytical solutions of this Problem have been reported by several authors during the last forty years. Danilovskaya [5] was the first to give closed-form, uncoupled, dynamic solutions for temperature and stress fields. The problem was later named after her as “First Danilovskaya Problem”. Besides sudden heating, Sternberg and Chakravorty [6] considered a more realistic ramp-type surface heating condition and gave closed-form solutions for displacement field, too. In literature the ramp-type heating case is usually named “Sternberg-Chakravorty Problem”. Within the framework of the classical, coupled theory (model 3), the Danilovskaya Problem was studied among others by Boley and Tolins [7], Wilms [8], and Ziegler [9]. Employing different methods, they gave approximate solutions and determined the wave speeds and discontinuities in temperature and stress.

Table 1. Most frequently used thermoelastic models

<table>
<thead>
<tr>
<th>Serial number</th>
<th>Name of the model</th>
<th>Equation of motion</th>
<th>Heat-conduction equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>classical, quasi-static</td>
<td>(4)</td>
<td>(5)</td>
</tr>
<tr>
<td>2.</td>
<td>classical, dynamic</td>
<td>(1)</td>
<td>(5)</td>
</tr>
<tr>
<td>3.</td>
<td>classical, coupled</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>4.</td>
<td>generalized, quasi-static</td>
<td>(4)</td>
<td>(6)</td>
</tr>
<tr>
<td>5.</td>
<td>generalized, dynamic</td>
<td>(1)</td>
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