A CLASS OF PERIODIC JACOBI-PERRON ALGORITHMS IN
PURE ALGEBRAIC NUMBER FIELDS OF DEGREE \( n \geq 3 \)

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The author obtains the periodicity of the Jacobi-Perron algorithm of \( (0, \ldots, 0, n-1) \) where \( 0^n = (D^n - d) / V^n \), with \( D, d, V \in \mathbb{N}^* = \{1, 2, \ldots\} \), \( d \mid D \), \( D \) and \( d \) congruent to \(+1(\mod V^{n-1})\) and \( D \equiv (n-1)d(V+1)/2 + 1 \). The case \( V = 1 \) has been studied by L. Bernstein and the proof for arbitrary \( V \) follows exactly the same pattern. Secondary results are then obtained from the main theorem.

0. INTRODUCTION

The Jacobi-Perron Algorithm (JPA) is one of the generalizations of the ordinary continued fraction algorithm to higher dimensions [1], and its definition is recalled at the end of this introduction.

Consider pure algebraic number fields of the form \( K = \mathbb{Q}(\omega) \) where

\[
\omega^n = M = D^n + d > 2 \quad (n \geq 3),
\]

with \( D, d \in \mathbb{N}^* = \{1, 2, \ldots\} \), and

\[
d \mid pD \quad \text{for} \quad n = p^n \quad (p \text{ prime}) \quad \text{or} \quad d \mid D \quad \text{for} \quad \text{other} \quad n.
\]

When an \( n \)th-power \( V^n \) divides \( M \), let

\[
0^n = m = M / V^n = \omega^n / V^n.
\]

L. Bernstein [1] showed that the JPA of a \((0) = (\omega, \ldots, \omega^{n-1})\) is periodic when

\[
M = D^n + d, \quad \text{with} \quad D \geq (n-2)d,
\]

\[
M = D^n - d, \quad \text{with} \quad D \geq 2(n-1)d.
\]

Using his methods and ideas described in [2], we obtain
the main result of this paper, viz. the periodicity of the JPA of $a^{(0)} = (\theta, \ldots, \theta_{n-1})$ when

$$M = D^n - d, \quad \theta^n = M/V^n,$$
with $D > (n-1)d(V+1)/2+1$, $D, d \equiv 1 \pmod{V^{n-1}}$ and $\theta \in \mathbb{N}^*.$

When $V = 1$ in (0.6), note the new bound for $D$.

Let $t, V \in \mathbb{N}^*$. L. Bernstein [5] obtained the periodicity of the JPA of $a^{(0)} = (\theta, \theta^2)$ when

$$M = (tV^3+1)^3 - 1, \quad \theta^3 = M/V^3 = t^3V^6 + 3t^2V^3 + 3t,$$
after having previously considered in [4] the case $t = V^3 + 2$ in (0.7). It is worth noting that if $D = tV^n+1$ and $d = 1$ in (0.6), we generalize L. Bernstein's case (0.7) to arbitrary dimension. Moreover when $V = nV_0$, $D = n^n - tV^n + 1$ and $d = 1$ in (0.6), we generalize the case $\theta^3 = ((9tV^3+1)^3 - 1)/27V^3 = 27t^3V^6 + 9t^2V^3 + tV_0$ previously studied in [6].

Let $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-1}^{(0)})$ be a vector of the real Euclidean vector space $\mathbb{R}^{n-1}$, $n \geq 2$. A sequence $<a(v)>$ of vectors of $\mathbb{R}^{n-1}$ is called the JPA of $a^{(0)}$ if for all $v \in \mathbb{N}$,

$$a(v+1) = \left(\frac{a_2^{(v)} - b_2^{(v)}}{a_1^{(v)} - b_1^{(v)}}, \ldots, \frac{a_{n-1}^{(v)} - b_{n-1}^{(v)}}{a_1^{(v)} - b_1^{(v)}}, \frac{1}{a_1^{(v)} - b_1^{(v)}}\right).$$

$$a_1^{(v)} = b_1^{(v)}, \quad b_1^{(v)} = \left[\frac{a_1^{(v)}}{a_1^{(v)} - b_1^{(v)}}\right] \quad (i = 1, \ldots, n-1),$$

where $[\cdot]$ is the greatest integer function.

The JPA of $a^{(0)}$ is called periodic, if there exist two integers $\ell, m$ with $\ell \geq 0$, $m \geq 1$ such that

$$a^{(v+m)} = a^{(v)} \quad (v = \ell, \ell+1, \ldots).$$

The sequences

$$a^{(0)}, a^{(1)}, \ldots, a^{(\ell-1)}$$
and

$$a^{(\ell)}, a^{(\ell+1)}, \ldots, a^{(\ell+m-1)}$$
are called respectively the preperiod and the period of the periodic JPA, and $\ell$ and $m$ are their respective lengths. When $\ell$ and $m$ are minimal, the preperiod and the period are said to be primitive. If $\ell = 0$, the JPA of $a^{(0)}$ is said to be purely periodic.

Let $\mathbb{N}_n = \mathbb{N}(a_1^{(0)}, \ldots, a_{n-1}^{(0)})$. If the JPA of $a^{(0)}$ becomes