Structured Fluid Theory

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With 2 Figures

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Summary — Zusammenfassung

Structured Fluid Theory. The simplest case of structured fluid theory is shown to correspond to a type of Maxwell fluid which exhibits normal stress differences. A nonlinear problem related to the deformation of suspended liquid drops as induced by inertial effects is solved and the influence of structured fluid parameters investigated.

Theorie strukturierter Flüssigkeiten. Es wird gezeigt, daß der einfachste Fall einer strukturierten Flüssigkeit einer Art Maxwellscher Flüssigkeit entspricht, welche Differenzen im Betrag der Normalspannungen aufweist. Ein nichtlineares Problem, welches mit der durch Trägheit verursachten Deformation eines Tropfens zusammenhängt, wird gelöst. Der Einfluß der Parameter der strukturierten Flüssigkeit wird untersucht.

Introduction

Many analyses of structured fluids have been given since the underlying works of Dahler and Scriven [1] and Eringen [2] appeared some ten years ago. We have been particularly interested in a special case which appears applicable to real flow situations. Considering the fluid structure to be spherical when undeformed, and thereafter to experience small deformations allows one to describe low shear rate flows of certain viscoelastic fluids (dilute solutions of random coiling macromolecules [3]) and also of dilute suspensions of liquid drops [4].

The purpose of the present work is twofold: to observe that the lowest order approximation of the theory applied to viscoelastic fluids produces a type of Maxwell fluid with normal stress effects; to solve the nonlinear problem involving deformations of suspended liquid drops, taking account of inertial effects.

Governing Equations

The equations governing the motion of structured fluids and the relation to analysis in classical fluid mechanics were discussed at some length by Allen, Kline, and LING [5], with numerous references given. We merely recall here that, in the absence of body forces and couples, the fluid and structure momentum equations are

\[ t_{ij,j} = \rho \dot{\theta}_i, \]
\[ \mu_{kij,k} + t_{ji} - m_{ji} = \rho (\dot{I}_{jk} W_{ki} - I_{kn} W_{ki} W_{nj}), \]
where \( \rho \) is the mass density, \( \mathbf{v} \) the local velocity, \( \mathbf{t} \) the stress tensor, \( \mathbf{m} \) the double stress tensor, \( \mathbf{m} \) the symmetric double force tensor and \( \mathbf{I} \) a measure of the microstructure moment of inertia. Further, if the fluid substructure is assumed to have a spherical shape in the undeformed state, then substructure deformations may be accounted for by means of a second order symmetric tensor \( \sigma \), defined by \text{KL\textsc{line}} and \text{AL\textsc{LEN}} [6] through the expression

\[
I_{ij} = \rho^2 (\delta_{ij} + \sigma_{ij}), \quad (3)
\]

where \( \rho \sqrt{2} \) is the volume averaged radius of gyration of the spherical substructure. The components of \( \sigma \) are restricted by the assumption of small deformations:

\[
|\sigma_{ij}| \ll 1; \quad i, j = 1, 2, 3. \quad (4)
\]

We shall consider incompressible fluids with volume preserving substructure. Then, making use of the fluid continuity equation, the kinematical constraints are

\[
v_{k,k} = 0, \quad \sigma_{kk} = 0. \quad (5)
\]

Moreover, applying the small deformation assumption (4), the substructure continuity equation reduces to [5]

\[
2W_{(ij)} = \delta_{ij} - \sigma_{ik} W_{[kj]} - \sigma_{jk} W_{[ki]}, \quad (6)
\]

in which symmetrization and antisymmetrization of enclosed indices are indicated by parentheses and brackets, respectively.

Constitutive equations for the stress tensor \( \mathbf{t} \), the relative stress \( \mathbf{t} - \mathbf{m} \), and the symmetric and anti-symmetric parts of the double stress tensor \( \mathbf{m} \), consistent with the linear theory, are taken from \text{KL\textsc{line}} and \text{AL\textsc{LEN}} [6] as:

\[
t_{ij} = -P \delta_{ij} + 2 \mu v_{(i,j)} - 2 \mu_1 (v_{[j,i]} + W_{[ij]}) - 2 \lambda_1 W_{(ij)}, \quad (7)
\]

\[
\mu_{k(ij)} = \lambda_2 (\xi_2 - \xi_1) \sigma_{ij,k}
\]

\[+ \frac{1}{2} (\lambda_3 \xi_1 + \lambda_4) (\delta_{ik} \sigma_{jp} + \delta_{jk} \sigma_{ip})
\]

\[+ \frac{1}{2} (\lambda_3 \xi_1 - \lambda_4) (\sigma_{ik,j} + \sigma_{jk,i})
\]

\[+ \xi_6 \sigma_{kp} \delta_{ij} + \lambda_3 (\xi_3 - \xi_4) W_{(ij,k)}
\]

\[+ \frac{1}{2} (\lambda_6 \xi_5 - \xi_7) (\delta_{ik} W_{(jp)},p + \delta_{jk} W_{(jp)},p)
\]

\[+ \xi_8 W_{(kp),p} \delta_{ij}.
\]

\[
\mu_{k[ij]} = n_1 W_{[ij,k]} + n_2 (W_{[k],j} - W_{[k],i})
\]

\[+ n_2 (\delta_{kj} W_{[ij],n} - \delta_{ki} W_{[ij],n}),
\]

(10)