Steady convection in a porous layer with translational flow

H. I. Ene and D. Poliševski, Bucharest, Romania

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Summary. We prove that if in a porous layer there is a translational flow in a horizontal direction, then the only stationary convection which may occur, in the presence of a small transversal gradient of temperature, is a helical motion with axes parallel to the translational flow. The stability analysis of this finite-amplitude steady solution is investigated, in the vicinity of the critical point.

1 Introduction

It is well known that a temperature gradient opposite to the direction of gravity in a horizontal porous layer saturated with fluid gives rise to a periodic motion, named convection. This effect was studied for the first time by Lapwood [1] and after that, different aspects of the problem were investigated. An excellent review of the problem may be found in Combarnous and Bories [2].

But in all the papers published in this field the existence of a transverse flow was neglected. In practical problems like groundwater motion, packed bed reactors, thermal oil recovery, such a flow exists and the interaction with the vertical temperature gradient may be investigated. The only result in literature about this problem is that of Prats [3] who proved that by changing the variable x into $\xi = x - bt$ the system of equations is transformed into the classical system of Lapwood, and consequently all known results can be immediately applied. For instance the criterion for the onset of convection remains $Ra_c = \alpha^2$ and it is independent of the Péclet number. But this solution is in fact an unsteady one.

In the present paper we consider only the steady convective motion and we prove that the helical motion with axes parallel to the translational flow is the only motion which may occur. Also we prove that any infinitesimal disturbances of the motion leads to stability. We have to mention that these results are in agreement with the experimental works (see Combarnous and Bories [2]) except the case of low Péclet numbers (less than 0.75) when the observed convective cells behave like rolls running perpendicular to the direction of flow.

More precisely if we consider a horizontal porous layer bounded by two planes maintaining a temperature gradient opposite to the direction of gravity, the dimensionless form of the system of equations may be written (see Ene and Poliševski [4]) as:

$$\text{div} \mathbf{V} = 0 \quad \text{for} \quad z \in (0, 1) \quad (1.1)$$

$$\mathbf{V} + \mathbf{V}_p = aTe \quad \text{for} \quad z \in (0, 1) \quad (1.2)$$

$$\partial_t T + \mathbf{V} \cdot \nabla T = \Delta T \quad \text{for} \quad z \in (0, 1) \quad (1.3)$$
\[ V \cdot e = 0, \quad T = 1 \quad \text{for} \quad z = 0 \quad (1.4) \]
\[ V \cdot e = 0, \quad T = 0 \quad \text{for} \quad z = 1 \quad (1.5) \]
\[ V \text{ and } T \text{ bounded as } (x, y) \rightarrow \infty \quad (1.6) \]

where \( e = (0, 0, 1) \), \( V \) is the Darcy’s velocity, \( p \) the pressure, \( T \) the temperature, \( \partial_t = \partial / \partial t \), \( \mathbf{V} = (\partial_x, \partial_y, \partial_z) \), and the Rayleigh number is defined by:
\[
Ra = \frac{k \rho c_p \alpha H \alpha (T_1 - T_2)}{v \zeta_m}. \quad (1.7)
\]

Here \( k \) is the permeability of the medium, \( T_1 > T_2 \) are the two temperatures of the impervious walls, \( \alpha \) is the thermal expansion coefficient, \( \zeta_m \) is the thermal conductivity of the porous medium, \( v \) is the kinematic viscosity of the fluid, \( H \) is the real thickness of the layer and \( c_f \) is the specific heat of the fluid.

If \( V_0 \) is the velocity of the translational flow in physical variables, we can introduce also the Péclet number
\[
b = \frac{\rho c_p \mu H V_0}{\zeta_m}. \quad (1.8)
\]

Now supposing that the temperature varies slowly on each stream line, we consider the uniform translational flow (which corresponds to the pure-conduction pattern) as the basic solution of (1.1)--(1.6). Then for \( b > 0 \) we have
\[ V_b = (b, 0, 0) \quad (1.9) \]
\[ T_b = 1 - z \quad (1.10) \]
and the pressure \( p_b \) obtained from (1.2).

Introducing \( \mathbf{U} = (u, v, w), \quad q \) and \( S \) by
\[ \mathbf{U} = \mathbf{V} - \mathbf{V}_b, \quad q = p - p_b, \quad S = T - T_b \quad (1.11) \]
the system (1.1)--(1.6) becomes
\[
div \mathbf{U} = 0 \quad z \in (0, 1) \quad (1.12) \]
\[
\mathbf{U} + Vq = Ra S \mathbf{e} \quad z \in (0, 1) \quad (1.13) \]
\[
\partial_t S + b \partial_x S + \mathbf{U} \cdot \nabla S = w + \Delta S \quad z \in (0, 1) \quad (1.14) \]
\[
w = S = 0 \quad z = 0, 1 \quad (1.15) \]
\[ \mathbf{U} \text{ and } S \text{ are bounded as } (x, y) \rightarrow \infty. \quad (1.16) \]

2 The finite-amplitude steady solutions

It is obvious that (1.12)--(1.16) with \( \partial_t = 0 \) represents a characteristic-value problem. As we want to study the convective motions which appear in the Rayleigh number regime near the critical point, we assume that the amplitude \( \varepsilon > 0 \) of the eigenfunction \( S \) defined by
\[
\varepsilon^2 = \sup_{(x,y) \in \mathbb{R}^3} \int_0^1 S^2 \, dz \quad (2.1)
\]