The stability of Hartmann flows for arbitrary magnetic Reynolds numbers is investigated in the framework of linear theory. The initial three-dimensional problem reduces to the equivalent two-dimensional problem. Perturbation theory is used to find asymptotic expressions for the eigenvalues. Distinguishing two types of disturbances — magnetic and hydrodynamic — is shown to be advantageous in a number of cases. Simple features of the stability are considered for particular cases. The well-known Lundquist result is generalized. An energy approach is applied to the problem of stability. The results of simulations involving the solution of the linear stability problem are described. A distinctive picture of stability is developed. There are several types of instability and they can develop simultaneously. The hydrodynamic and magnetic phenomena interact with each other in a very complex fashion. The magnetic field can either enhance flow stability or reduce it.

1. The stability of the flow of a viscous incompressible conducting fluid between parallel, ideally conducting plates perpendicular to a homogeneous magnetic field is investigated. The problem in linear theory reduces to the search for the eigenvalues of the following problem [1]:

\[
(U - c) (\psi'' - k^2 \psi) - U'' \psi + \frac{i}{\alpha R m} (\psi'' - 2k^2 \psi'') + k^4 \psi = \frac{G^2}{R m} (H_x (\Phi'' - k^2 \Phi) - \frac{i}{\alpha} (\Phi'' - k^2 \Phi')) - H_x' \Phi \]
\]

where \( U \) is the velocity profile and \( H_x \) is the intensity profile of the longitudinal magnetic field; \( \psi \) and \( \Phi \) are complex amplitudes of the transverse perturbation components of the velocity and the field; \( c = X + iY \) is the complex phase velocity of the disturbances (the eigenvalue of the problem); \( k \) is the wave vector modulus, \( \alpha = k_x \); \( R \) is Reynolds number; \( R_m \) is the magnetic Reynolds number; and \( G \) is the Hartmann number.

The values \( Y < 0 \) correspond to decay of the disturbances and \( Y > 0 \) to growth. The coordinate system is chosen so that the \( x \) axis is parallel to the flow and the \( y \) axis is perpendicular to the plates. The characteristic length is the half-width \( L_0 \) of the channel, the characteristic velocity is the flow velocity \( V_0 \) at the axis of the channel, and the characteristic magnetic field intensity is the intensity \( H_0 \) of the external magnetic field.

The problem of the modified Orr-Sommerfeld equation with homogeneous boundary conditions for \( \psi \) and \( \psi' \) was studied in detail in [1,2] for small \( R_m \). This eigenvalue problem was considered in detail in [3-5] for a number of different \( U \).

Different considerations were put forth in [6,7] with respect to the stability of Hartmann flows for arbitrary \( R_m \). Below, an attempt is made to obtain a fairly complete picture of stability for arbitrary \( R_m \).
2. We reduce the three-dimensional problem to the equivalent two-dimensional problem through transformations similar to those performed by Hunt for the case of a homogeneous longitudinal magnetic field [8].† Therefore, we set

\[ aR = kR, \quad aR_m = kR_m, \quad a\Phi = \Phi \]  

(2.1)

in the initial equations.

If \( R_* = f(R_m, G) \) is determined, the behavior of \( R_* = f(R_m, G) \) can be analyzed without additional calculations. Let \( \theta \) be the angle between the direction of the wave vector \( k \) and the flow direction. Then

\[ R_* = \min \frac{f(R_m \cos \theta, G)}{\cos \theta} \]

Henceforth, we will consider the equivalent two-dimensional problem. The tilde will be dropped and we set \( k = \alpha \) in (1.1).

It is useful to determine the asymptotic expressions for \( c_n \). We consider the case of small \( \alpha R \) and \( \alpha R_m \). Using perturbation theory, we seek the solution in the form

\[ \Phi = \varepsilon \Phi_0 + \varepsilon^2 \Phi_1 + \ldots, \quad \psi = \psi_0 + \varepsilon \psi_1 + \ldots, \quad c = \varepsilon^{-1} c_0 + c_1 + \varepsilon c_2 + \ldots \]

where \( \varepsilon = \alpha R \). Also, we consider \( \alpha^2 \) to be small. Then, in the zeroth approximation we obtain

\[ \Phi_0'' + \varepsilon \Phi_0' = -\delta \psi_0', \quad \psi_0'' + i \varepsilon \psi_0' = -\left( G^2 / \delta \right) \Phi_0'' \quad (-1 < y < 1) \]

\[ \psi_0(\pm 1) = \psi_0'(\pm 1) = 0, \quad (\delta = R_m / R) \]

(2.2)

where \( \delta \) is the magnetic Prandtl number. The zeroth approximation is equivalent to considering a quiescent fluid. The latter problem was studied in [10]; however, only symmetric (with respect to \( \psi \)) solutions were considered. In this case the denumerable set of eigenvalues is determined by

\[ c_n^{(0)} = -i (\alpha R)^{-1} \left[ \frac{(1 + \delta)^2} {8\delta} \pm \frac{\pi(n + 1)} {2\sqrt{\delta}} \right] \sqrt{\frac{(1 - \delta)^2}{16\delta} \pi^2(n + 1)^2 - G^2} \]

(2.3)

where \( c_n^{(0)} = c_0 n(\alpha R)^{-1} \). The analysis of perturbations antisymmetric with respect to \( \psi \) reduces to the search for the roots of a complex transcendental equation

\[ \cosh k_1 \cosh k_2 \left[ k_1^2 (\sinh k_1 - k_1^2 - k_2^2 (\sinh k_1 - k_1^2) - \frac{\lambda}{1 + \delta} (k_1 k_2 - k_1 \sinh k_2) \right] = 0 \]

\[ k_{1,2} = \sqrt{\frac{1}{2} (\lambda - G^2) \pm \sqrt{1/4 (\lambda - G^2)^2 - a \lambda}} \quad \lambda = i c_0 (1 + \delta) \quad a = \delta (1 + \delta)^{-2} \]

(2.4)

In the general case, numerical calculations are needed to find the roots of (2.4).

We analyze the expression (2.3) obtained for \( c_n \). When the term under the sign of the radical is positive, the disturbances decay monotonically. With an increasing Hartmann number the term under the sign of the radical can become negative, which corresponds to the appearance of oscillatory disturbances even in the zeroth approximation. (Similar effects were analyzed in [11] for the case of a quiescent fluid in a cavity of fairly arbitrary shape.) The presence of oscillating disturbances is associated with finite fluid conductivity. They vanish when \( \delta \to 0 \) (for a fixed value of \( G \)). It is clear that here the phase velocity can be larger than unity, in contrast to problems with small \( R_m \). As expected, the term \( Y \) in (2.3) is always negative.

When \( G \to 0 \) in (2.3), we have

\[ c_n^{(0)^-} = -i \frac{\pi^2(n + 1)^2}{4 \alpha R}, \quad c_n^{(0)^+} = -i \frac{\pi^2(n + 1)^2}{4 \alpha R_m} \]

(2.5)

† The case of a coplanar magnetic field was considered by a number of other authors, in particular, [9].