Note

One more general method for solving Oseen's equations

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Summary. In this paper we have proposed a new method for solving the Oseen's equations. It has been testified in the case of the normal flow around the cylinder for the Re number being much smaller than 1. The step forward was to generalize this problem and find out the solution in the case of the oblique flow around the cylinder. We have also applied the method to treat the normal flow around the cylinder in the range of Re numbers comparable to unity. The obtained coefficients of the resistance have also been presented in the figures at the end of the paper.

1 Introduction

Oseen's equations and the equation of continuity for an incompressible flow are known:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v}, \quad \text{div} \mathbf{v} = 0,$$

where $U$ is the velocity far from the object of interest in $x$ direction and $\rho$, $p$, $\mathbf{v}$, $\nu$ being density, pressure, velocity and kinematic viscosity respectively. The solution of the Eqs. (1) is known in the case of perpendicular flow around the cylinder and sphere, but it has been derived starting from the assumption that the vortex lies in the plane perpendicular to the velocity $U$ [1]. That solution is:

$$\mathbf{v} = \nabla \varphi + \frac{1}{2\lambda} \nabla \chi - \chi \mathbf{i},$$

where $\varphi$ and $\chi$ are the solutions of the equations:

$$\Delta \varphi = 0, \quad \Delta \chi - 2\lambda \frac{\partial \chi}{\partial x} = 0, \quad \lambda = U/2\nu.$$

The pressure $p$ satisfies the equation:

$$p = p_0 - \rho U \frac{\partial \varphi}{\partial x},$$

where $p_0$ denotes its value far enough from the object of interest. As we have already seen, this solution has some limits and is not the general one.

Our purpose is to treat the Oseen's equations in a more general manner. Starting from the idea of decomposing velocity into two parts, we assume:

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \nabla \varphi + \mathbf{v}_2$$

(2)
having no other constraints on the velocity \( v_2 \). If the potential \( \varphi \) and the pressure \( p \) satisfy the same equations as before,

\[
\Delta \varphi = 0, \quad p + \varphi U \frac{\partial \varphi}{\partial x} = p_0,
\]

the system (1) becomes:

\[
U \frac{\partial v_2}{\partial x} = v_2 \Delta v_2, \quad \text{div} \, v_2 = 0. \tag{4}
\]

In this way the problem is reduced to the solution of a single Laplace’s equation \( \Delta \varphi = 0 \) and the system (4). The equation \( \text{div} \, v_2 = 0 \) implies:

\[
v_2 = \text{rot} \, A = -\text{grad} \, f_1 \times \text{grad} \, f_2, \tag{5}
\]

with the vector potential \( A \), while \( f_1 \) and \( f_2 \) represent the stream surfaces of the velocity \( v_2 \).

It is obvious from the previous equation that the vector \( A \) is determined up to a \( \text{grad} \, \beta \) \((f_2 \text{ being any function})\). That means, that in the complete treatment we can insert \((A + \text{grad} \, f_2)\) instead of \( A \) without changing any physics. One possibility is to choose the function \( f_2 \) so that \( \text{div} \, A = 0 \) is satisfied. That also implies:

\[
2A = f_2 \text{ grad} \, f_1 - f_1 \text{ grad} \, f_2 + \text{ grad} \, \beta, \quad \Delta f_2 = f_1 \Delta f_2 - f_2 \Delta f_1.
\]

After imposing \( \Delta A = -\text{rot} \, \text{rot} \, A = -2\omega \) \((\omega \text{ — vorticity})\) and \( \Delta v_2 = -\text{rot} \, \text{rot} \, v_2 = \text{rot} \, (\Delta A) \), we get the first equation of the system (4) in the following form:

\[
\text{rot} \left( \frac{U}{v} \frac{\partial A}{\partial x} - \Delta A \right) = 0. \tag{6}
\]

The Eq. (6) is fulfilled in two cases:

\[
\frac{U}{v} \frac{\partial A}{\partial x} - \Delta A = 0 \quad \text{or} \quad \frac{U}{v} \frac{\partial A}{\partial x} - \Delta A = \text{grad} \, \phi, \tag{7}
\]

where the function \( \phi \) is the solution of the Laplace’s equation, having in mind that

\[
\text{div} \left( \frac{U}{v} \frac{\partial A}{\partial x} - \Delta A \right) = 0
\]

is also satisfied. According to that it is necessary to solve one or two Laplace’s equations \((\Delta \varphi = 0, \Delta \phi = 0)\) and the system of three Eqs. (7).

\section*{2 Perpendicular flow around the cylinder}

The suggested method has been testified in the case of the flow around the cylinder with the velocity being perpendicular to its axis \((z\text{-axis})\). The solution is already known and we could compare our result with it. The entire treatment we perform in the cylindrical coordinate system taking into account only the first harmonics. The first equation in the system (3) has the following solution:

\[
\varphi \approx B_s \ln r + B_1 \frac{\cos \theta}{r} + B_2 \frac{\sin \theta}{r}. \tag{8}
\]