A Note on a Stability Problem in Hydrodynamics

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With 1 Figure

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Summary — Zusammenfassung

A Note on a Stability Problem in Hydrodynamics. The initial-value problem corresponding to a perturbed heterogeneous shear flow is considered. When the normal-mode method is applied, the solution of the problem is most easily obtained by introducing the concept "the finite part of an integral". In fact it means that the eigen-functions corresponding to the continuous spectrum of eigenvalues are generalized functions rather than ordinary functions.

The problem is also solved by the Laplace transform technique, and the equivalence between the solution obtained by this method and the one obtained by the method of normal modes is demonstrated.

1. Introduction

In this note we are concerned with a stability problem in hydrodynamics. We consider the initial-value problem which arises in connection with the study of the stability of the Couette flow in a stratified fluid between two rigid horizontal planes. It is known that in this and related problems in hydrodynamics the eigenfunctions corresponding to the discrete spectrum of eigenvalues, which is obtained when the normal-mode method is applied, do not form a complete set. In many situations there is only a finite number of discrete eigenvalues (see for instance [1], [2], [3]). The most simple example of such a situation is the Couette flow in a homogeneous fluid, where there is no discrete spectrum at all. Consequently an arbitrary function which is taken to be an initial-value in the problem, cannot be expanded in terms of the eigenfunctions corresponding to the discrete spectrum only. We also have to take into account the eigenfunctions corresponding to the continuous spectrum of eigenvalues, which is always present in these problems.
The problem we are considering has previously been investigated in [1], where the normal-mode method is applied to solve the initial-value problem. In [1] it is given explicit solution to the problem only in the two cases when \( r > \frac{1}{4} \) and \( 0 < r < \frac{1}{4} \), \( r \) is the dimensionless Richardson number. In the case \( r > \frac{1}{4} \) there is an infinite set of discrete eigenvalues, and it is assumed in [1] that the eigenfunctions corresponding to this discrete spectrum form a complete set. This assumption is not valid. In the case \( 0 < r < \frac{1}{4} \) there is no discrete spectrum of eigenvalues at all.

In this note we indicate how to solve the problem for all values of \( r \) by the normal-mode method. In the case \( 0 < r < \frac{1}{4} \) our solution is the same as the one given in [1]. Both the discrete and the continuous spectrum of eigenvalues have to be taken into account when the normal-mode method is applied. In general our solution can be represented as a sum of two parts; one part is the sum over the eigenfunctions corresponding to the discrete spectrum, the other part is the integral over the eigenfunctions corresponding to the continuous spectrum. The continuous spectrum consists of the range of \( U(z) \), where \( U(z) \) is the unperturbed parallel flow velocity, varying in the direction perpendicular to the flow direction.

If the eigenfunctions corresponding to the continuous spectrum of eigenvalues are taken to be generalized functions, (see [4], [5]), our problem is easily solved for all values of \( r \) by the normal-mode method. In section 2 we introduce the concept “the finite part of an integral”, (see [5]), and solve the initial-value problem. To tackle the problem in this manner has not yet been done as far as we know.

In section 3 we solve the problem by the Laplace transform technique, which is a very simple and direct method. It is shown that the solution obtained by this method is equivalent to the one obtained by the normal-mode method when both the discrete and continuous spectrum of eigenvalues are taken into account.

We are concerned with the stability of the following system: a plane parallel flow of a stratified, incompressible and inviscid fluid confined between two rigid horizontal planes. The velocity field and the density field in the basic motion are:

\[
\begin{align*}
U &= \alpha z, \\
\rho &= \rho_0 e^{-\beta z},
\end{align*}
\]

where \( \alpha, \rho_0, \beta \) are constants.

The basic flow is along the \( x \)-axis and varies in the \( z \)-direction perpendicular to the flow direction.

We formulate our stability problem as an initial-value problem. The basic motion (eq. (1.1)) is assumed to be perturbed at a given point of time, and the subsequent motion is studied. The equations governing the motion are the hydrodynamic equations for a stratified, incompressible and inviscid fluid. These equations are linearized, assuming the perturbation in the velocity field and the density field to be small compared with the respective quantities in the basic motion. Applying the Boussinesq approximation, and using \( \alpha^{-1} \) as the unit of