Note

Unsteady Couette flows of second grade fluids in heated cylindrical domains

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Summary. By means of finite Hankel transforms, the exact solutions corresponding to some unsteady flows of second grade fluids in heated cylindrical domains are obtained. For steady flows, both the velocity field and the temperature are identically to those resulting from the Navier-Stokes theory.

1 Introduction

For the description of the mechanical or thermomechanical behavior of non-Newtonian fluids, much work has been devoted to the study of second grade fluids, they are of interest for theory and experiments. For these fluids the Cauchy stress $T$ and the fluid motion are related by [1]

$$ T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \quad (1.1) $$

where $A_1$ and $A_2$ are the first two Rivlin-Ericksen tensors, $\mu$ is the viscosity, $\alpha_1$ and $\alpha_2$ are normal stress moduli, and $-pI$ denotes the indeterminate spherical stress due to the constraint of incompressibility. The constitutive assumption (1.1) can be considered to be the second order approximation to the response functional of a simple fluid in the sense of retardation [2]. On the other hand, since the constitutive model is properly invariant it could be thought as an exact model.

In the following we consider a flow whose velocity field, in the cylindrical coordinate system $r, \varphi$ and $z$, has the contravariant components

$$ \dot{r} = 0, \quad \dot{\varphi} = \omega(r,t), \quad \dot{z} = 0, \quad (1.2) $$

and the temperature field is of the form

$$ \theta = \theta(r,t). \quad (1.3) $$

By virtue of (1.1) and of Fourier’s law of heat conduction,

$$ q = -k \text{grad} \theta, \quad (1.4) $$
where $q$ is the heat flux vector and $k$ is the conductivity, the momentum and the energy equations reduce to [1], [3]

\[(\mu + \alpha_1 \partial_t) \left( \partial_r^2 + \frac{3}{r} \partial_r \right) \omega(r, t) = q \partial_t \omega(r, t) \]  

(1.5)

and, respectively,

\[\rho c \partial_t \theta(r, t) = k \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \theta(r, t) + \mu \nu^2 (\partial_r \omega(r, t))^2 + q h(r, t), \]

(1.6)

where $\rho$ is the density of the fluid, $c$ is the specific heat, $h(\cdot, \cdot)$ is the radiant heating or the heat supply per unit mass, and the subscripts $r$ and $t$ indicate the partial differentiation with respect to the two arguments.

**Remark 1.** Looking to Eqs. (1.5) and (1.6) we can say that the motion of the fluid is not influenced by its temperature, although the opposite is not true.

## 2 Flow between heated cylinders

The flow between rotating cylinders is one of the most important and most interesting problems of the motion near rotating bodies [4]. During recent years quite a lot of papers on this type of flow had been published.

Consider a second grade fluid at rest in the annular region between two infinitely long co-axial cylinders of radii $R_1$ and $R_2$ ($R_2 > R_1$), (see Fig. 1). At time $t = 0$, both cylinders suddenly start rotating about their axis, with the angular velocities $\Omega_1(t)$ and $\Omega_2(t)$.

Assuming that the two cylinders have the temperatures $\theta_1(t)$ and $\theta_2(t)$ and the fluid sticks to the walls, and making the changes of unknown functions

\[\begin{align*}
\omega(r, t) &= \Omega_2(t) + \frac{\Omega_2(t) - \Omega_1(t)}{R_2^2 - R_1^2} \cdot \frac{R_1^2 (r^2 - R_2^2)}{r^2} + \frac{1}{r} v(r, t) \\
\theta(r, t) &= \theta_2(t) + \frac{\theta_2(t) - \theta_1(t)}{\ln(R_2/R_1)} \ln(r/R_2) + u(r, t)
\end{align*}\]

(2.1)

and

\[\begin{align*}
(\mu + \alpha_1 \partial_t) \Delta v(r, t) + a(r, t) &= \rho \partial_t v(r, t); \quad r \in (R_1, R_2), \quad t > 0, \\
v(r, 0) &= V(r); \quad r \in (R_1, R_2), \\
v(R_1, t) &= v(R_2, t) = 0; \quad t \geq 0
\end{align*}\]

(2.3.1)

\[\begin{align*}
\rho c \partial_t \theta(r, t) &= k \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \theta(r, t) + \mu \nu^2 (\partial_r \omega(r, t))^2 + q h(r, t), \\
u(r, 0) &= U(r); \quad r \in (R_1, R_2), \\
u(R_1, t) &= u(R_2, t) = 0; \quad t \geq 0
\end{align*}\]

(2.3.2)

and

\[\begin{align*}
(\mu + \alpha_1 \partial_t) \Delta u(r, t) + b(r, t) &= \rho \partial_t u(r, t); \quad r \in (R_1, R_2), \quad t > 0, \\
u(r, 0) &= U(r); \quad r \in (R_1, R_2), \\
u(R_1, t) &= u(R_2, t) = 0; \quad t \geq 0.
\end{align*}\]

(2.3.3)