A Maximum-Dissipation Principle in Generalized Plasticity

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Summary

It is shown that in large-deformation generalized plasticity a local maximum-dissipation postulate is equivalent to the condition that the plastic strain rate (in the sense of Rice) cannot oppose the total strain rate, when strain space is regarded as a Riemannian manifold with the instantaneous Lagrangian tangent elastic stiffness as the metric tensor. From this condition, normality conditions in strain space (in this sense) and in the space of the second Piola-Kirchhoff stress (in the usual sense) are derived. With the additive decomposition of strain, the loading surface has essentially the same properties as in infinitesimal-strain plasticity. For the multiplicative decomposition, approximate normality rules are derived.

1. Introduction

The principle of maximum plastic dissipation, which in classical plasticity theory has the form

\[(\sigma_{ij} - \sigma^*_{ij}) \dot{\epsilon}_{ij} \geq 0,\]

was derived by Mandel [1] as the generalization to elastic-plastic solids of a postulate that, for rigid-plastic solids, was introduced originally by von Mises [2], but appears to have been derived independently by Hill [3]. In Eq. (1) \(\dot{\epsilon}_{ij}\) denotes the plastic strain-rate tensor, \(\sigma_{ij}\) the actual stress tensor, and \(\sigma^*_{ij}\) any stress tensor lying on or inside the current yield locus. Eq. (1) is an obvious generalization to an arbitrary state of stress and strain of the following inequality which is applicable to states with only one stress or strain component:

\[(\sigma - \sigma^*) \varepsilon^p \geq 0.\]

This inequality has the simple interpretation that the actual stress is greater (less) than any other stress attainable by unloading if the plastic strain rate is positive (negative). In fact, the derivation of Eq. (1) by all three authors is based on the
notion that Eq. (2) governs the shear stress and plastic shear rate on each slip system of a crystal, and the macroscopic inequality (1) is obtained from (2) by an application of the principle of virtual work; see Hill [4] or Mandel [5].

It was pointed out by Prager [6] that von Mises, Taylor [7] and Hill presented the principle only for a regular yield locus, and that its validity for singular yield locus relies on Koiter’s [8] generalization of the theory of the plastic potential. The principal consequences of Eq. (1) are the following [5]: (1) where the yield locus is regular, the plastic strain rate is perpendicular to it ("normality rule"); (2) where the yield locus forms a corner, the plastic strain-rate vector lies in the cone formed by the limiting normal vectors; (3) the yield locus in convex. One may combine these results by saying, in the language of Moreau [9], that the plastic strain rate belongs to the subgradient of the yield function.

It is obvious from Eq. (2) that the validity of the postulate is not limited to work-hardening materials; the only requirement is that the direction of the plastic strain rate in a loading process must oppose (that is, form an obtuse angle with) the change of stress in an unloading process. Consequently, stability in the sense of Drucker [10] is a stronger assumption than the principle of maximum plastic dissipation, and it is not difficult to show that Drucker’s local inequality,

$$\dot{\sigma}_{ij} \dot{\epsilon}_{ij}^p \geq 0$$

implies Eq. (1). A local inequality essentially equivalent to (1) was derived by Nguyen and Bui [11], namely,

$$\dot{\sigma}_{ij} \dot{\epsilon}_{ij}^p \geq -E_{ijkl} \dot{\epsilon}_{ij}^p \dot{\epsilon}_{kl}^p,$$

where $E_{ijkl}$ is the elastic stiffness tensor; a similar inequality had previously been derived by Ilyushin [12]. Eq. (4) may also be written as

$$E_{ijkl} \dot{\epsilon}_{ij}^p \dot{\epsilon}_{kl}^p \geq 0,$$

where $\dot{\epsilon}_{ij}$ is the total strain-rate tensor. The one-dimensional versions of inequalities (4) and (5) are, respectively,

$$\dot{\epsilon} \dot{\epsilon}^p \geq 0 \quad \text{and} \quad \dot{\epsilon} \dot{\epsilon}^p \geq 0.$$

They may consequently both be phrased as "the plastic strain rate cannot oppose the stress rate (total strain rate)", and both may be interpreted as stability postulates — the former being stability under stress control and latter being stability under strain control (or kinematic stability). It is an experimental fact that work-softening materials behave unstably in a stress-controlled test and stably in a strain-controlled test, provided they obey (2).

A remarkable property of inequality (1), and one which will be exploited here, is the fact it can be phrased in a way such that there is no mention of the yield locus; one need only define $\sigma_{ij}^*$ as the stress at any state that can be attained