Slip-Plane and Slip-Cylinder Theorems for Slow Viscous Flow

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With 2 Figures

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Summary — Zusammenfassung

Slip-Plane and Slip-Cylinder Theorems for Slow Viscous Flow. Theorems for the slow two-dimensional flow of viscous fluids about slip-planes and slip-cylinders are proven. Examples of the use of the theorems are given.

Gleitebenen- und Gleitzylinder-Theoreme für langsames viskoses Fließen. Theoreme für das langsame zweidimensionale Fließen von viskosen Flüssigkeiten um Gleitebenen und Gleitzylinder werden bewiesen und Beispiele für die Anwendung dieser Theoreme angegeben.

1. Introduction

In the theory of incompressible inviscid fluids Milne-Thomson [1] proved what became known as a Circle Theorem. It gives the complex potential for an unbounded flow (without rigid boundaries) disturbed by the introduction of a circular cylinder into the flow, in terms of the complex potential of the undisturbed flow. Later Levin [2] relaxed somewhat the condition that no rigid boundaries are present. A complete discussion of the Circle Theorem is given in [3].

Ionescu [4] gave an equivalent treatment for the slow flow of a viscous incompressible fluid. In this work we will prove slip-plane and slip-cylinder theorems for slow viscous flow. In other words, we will give a stream function for an unbounded flow without rigid boundaries disturbed by introduction of a slip plane or slip cylinder into the flow, in terms of the stream function of the undisturbed flow. By the slip plane or cylinder we mean the plane or cylinder on which the component of the stress vector in the tangential direction at any point of the plane or cylinder is zero. Physically, slip conditions on the plane are met in the flows through porous media while the slip cylinder may represent an air bubble with large surface tension introduced in a fluid flow [5]. Also, when a fluid does not wet a solid on which it moves it is more appropriate to use the boundary condition of zero shear stress than that of no-slip.

2. Equations of Motion and Boundary Conditions

For slow two-dimensional flow of a Newtonian fluid equations of motion reduce to

\[ \nabla^4 \psi = 0 \]
where \( \psi \) is a stream function, i.e., velocity components in the \( x \) and \( y \) directions of some fixed Cartesian coordinate system are given as \( u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x}. \)

Introducing the complex variable \( z = x + iy \) and putting \( \bar{z} = x - iy \), (1) becomes

\[
\frac{\partial^4 \psi}{\partial z^2 \partial \bar{z}^2} = 0. \tag{2}
\]

Integrating (2) one gets the general solution to the biharmonic Eq. (1) as:

\[
\psi = \text{Re} \{ \bar{z} f_1(z) + g(z) \} \tag{3}
\]

where \( \text{Re} \{ . \} \) denotes the real part of the expression in brackets, and where \( f_1(z) \) and \( g(z) \) are arbitrary analytic functions of \( z \).

If \( C \) represents a slip surface in the flow the following pair of boundary conditions have to be satisfied:

i) The velocity component normal to the surface is zero (i.e., the surface is a streamline).

ii) The component of the stress vector along the tangential direction of the surface is zero.

The velocity expressed in terms of the analytic functions \( f_1(z) \) and \( f_2(z) \) takes the form

\[
-v + iu = f_1(z) + z f_1'(z) + f_2(z) \tag{4}
\]

where \( f_1'(z) = \frac{d}{dz} f_1(z) \). Using (4) condition i) becomes:

\[
\text{Re} \left\{ \frac{d\bar{z}}{ds} \left[ f_1(z) + z f_1'(z) + f_2(z) \right] \right\} = 0 \tag{5}
\]

where \( z = z(s) \) denotes the parametric form of the slip surface, \( s \) being arc length. Also, the force exerted by the fluid on a fixed surface is

\[
X + iY = 2\mu \left\{ \left[ f_1(z) - \bar{f}_1(\bar{z}) \right] \frac{dz}{ds} - \left[ z f_1''(z) + \bar{f}_1'(\bar{z}) \right] \frac{d\bar{z}}{ds} \right\} \tag{6}
\]

so that condition ii) can be written as a requirement that the force given by (6) acts along the direction normal to the slip surface. We will later express this condition more specifically for the geometries considered.

Analytic functions \( f_1(z) \) and \( f_2(z) \) are sometimes called Goursat functions [6], [7].

3. Slip-Plane Theorem

Suppose \( f_1(z) \) and \( f_2(z) \) are Goursat functions of a steady plane Stokes flow. Suppose further that all singularities of \( f_1 \) and \( f_2 \) lie at a finite distance in the upper half plane. If a plane typified by the real axis becomes a slip surface in this flow, Goursat functions for the disturbed flow in the upper half plane become

\[
\hat{f}_1 = f_1 - \bar{f}_1(\bar{z}) \tag{7}
\]

\[
\hat{f}_2 = f_2 - \bar{f}_2(\bar{z})
\]