THE GENERAL STONE-WEIERSTRASS PROBLEM
AND EXTREMAL COMPLETELY POSITIVE MAPS

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Let $A$ be a C*-algebra with identity $e$ and let $B$ be a C*-subalgebra of $A$ that contains $e$. We show that if $B$ separates the pure states of $A$, then, for each $n$, $B$ also separates the set $ECP(A, C^n; I)$ of extremal completely positive unital maps of $A$ into $C^n$, thus giving another equivalent condition for the general Stone-Weierstrass conjecture for C*-algebras.

Let $B$ be a C*-subalgebra of a C*-algebra $A$ and assume that $B$ contains the identity $e$ of $A$. The general Stone-Weierstrass conjecture is that $B = A$ if $B$ separates the pure states of $A$. For summaries of the partial progress on this problem, see [1] or [4]. Since then, building on results from [2], Longo and Popa have proved, independently, in [7] and [8] that $B = A$ if $A$ is separable and $B$ separates the factor states of $A$.

In a recent paper [6], I. Fujimoto and S. Takahasi proved that several conditions are equivalent to $B$ separating the pure states of $A$. One such condition is that $B$ separate the set of nonzero pure completely positive maps from $A$ into $L(H)$ for $H$ a Hilbert space. In this note we consider the condition that $B$ separate the set of extremal completely positive unital maps of $A$ into $L(H)$ and show that if $H$ is finite-dimensional this condition is equivalent to $B$ separating the pure states of $A$. On the other hand, if $B$ separates the extremal completely positive unital maps of $A$ into $L(H)$ for every Hilbert space $H$, it follows
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that B = A. We also examine the two conjectures embedded in Conjecture 3 of [6] and show that one of them is equivalent to the general Stone-Weierstrass conjecture, and the other is false.

We use the concepts and notation of Arveson [3, Chapter 1]. For A a C*-algebra with identity, a completely positive linear map of A into L(H), the C*-algebra of bounded linear operators on H, has the form

\[ \phi(x) = V^{*} \pi(x)V \]

where \( \pi \) is a *-representation of A on some Hilbert space \( H_{\pi} \) and V is a bounded operator from H to \( H_{\pi} \). If \([\pi(A)VH]_{\pi}\), the closed linear span of \( \pi(A)VH \), equals \( H_{\pi} \), then \( V^{*} \pi(x)V \) is the unique (up to a unitary intertwining operator) such decomposition of \( \phi \) and is called the Stinespring decomposition of \( \phi \). Let \( \text{CP}(A,H) \) denote the set of completely positive linear maps of A into L(H). Let \( \text{CP}(A,H;I) \) denote the set of \( \phi \in \text{CP}(A,H) \) for which \( \phi(I) = I \), the identity of L(H). By [3, Theorem 1.4.6] \( \phi = V^{*} \pi V \) is an extreme point of the (compact) convex set \( \text{CP}(A,H;I) \) if and only if the map

\[ X \mapsto VV^{*}X|_{V(H)} \]

from \( \pi(A)' \) to L(V(H)) is one-to-one. A completely positive linear map \( \phi = V^{*} \pi V \) is called pure if \( \pi \) is irreducible and V \( \neq 0 \). Let \( \text{ECP}(A,H;I) \) denote the set of extreme points of \( \text{CP}(A,H;I) \). We say that a C*-subalgebra B of A separates \( \text{ECP}(A,H;I) \) if \( \phi = \theta \) whenever \( \phi \) and \( \theta \) are in \( \text{ECP}(A,H;I) \) and \( \theta|_{B} = \phi|_{B} \).

**Theorem 1.** If A is a C*-algebra with identity e and B is a C*-subalgebra of A which contains e and separates the pure states of A, then B separates \( \text{ECP}(A,\mathbb{C}^{P};I) \) for every natural number n.