THEOREMS OF THE ORLICZ-PETTIS-TYPE FOR LOCALLY CONVEX SPACES

Peter Dierolf

For a dual pair \( \langle E, F \rangle \) we determine the finest \( \langle E, F \rangle \)-polar topology on \( E \) which has the same subfamily-summable (respectively bounded multiplier summable) families as the weak topology \( \sigma(E, F) \). It is shown that these characterizations contain most of the known theorems of the ORLICZ-PETTIS-type for locally convex spaces and also several new results of this type. Some applications to spaces of functions and operators are given.

1. Introduction

Theorems of the ORLICZ-PETTIS-type concern the summability of families with respect to different topologies. We consider the following notions of summability

\[(1.1) \text{ DEFINITION.} \text{ Let } (E, \mathcal{R}) \text{ be a locally convex space, } (x_i; i \in I) \text{ a family in } E, \text{ and let } \mathcal{F}(I) \text{ denote the set of all finite subsets of } I \text{ which is directed by inclusion.}
\]

(a) \( (x_i; i \in I) \) satisfies the CAUCHY-condition (is summable), if the net \( (\sum_{i \in \sigma} x_i; \sigma \in \mathcal{F}(I)) \) is a CAUCHY-net (is convergent).

(b) \( (x_i; i \in I) \) is SF-summable (subfamily-summable) if \( (x_i; i \in J) \) is summable for all \( J \subset I \).

(c) \( (x_i; i \in I) \) is BM-summable (bounded-multiplier-summable) if \( (\lambda x_i; i \in I) \) is summable for all \( \lambda = (\lambda_i; i \in I) \in l^\infty(I) \).

A family \( (x_i; i \in I) \) satisfies the CAUCHY-condition in \( (E, \mathcal{R}) \) iff one of the following equivalent conditions is satisfied:

(a') For every O-nbd \( U \in \mathcal{U}_0(E, \mathcal{R}) \) there exists \( \sigma \in \mathcal{F}(I) \) with the property: \( \forall \tau \in \mathcal{F}(I), \tau \cap \sigma = \emptyset \iff \rightarrow \)
\[ (\sum_{i \in I} x_i) \in U. \]

(a") For every \( 0 \)-nbhd. \( U \in \mathcal{V}_0(E, \mathcal{K}) \) there exists \( G \in F(I) \) with the property: \( \tau \in F(I), \quad \tau \cap G = \emptyset \implies \sum_{i \in I} |\langle x'_i, x_i \rangle| \leq 1 \) for all \( x'_i \in U^0. \]

(cf. BOURBAKI [5, p. 61]). From these conditions it is easy to see that \((x_i: i \in I)\) satisfies the CAUCHY-condition iff every countable subfamily satisfies the CAUCHY-condition.

The following theorem is a locally convex version of the classical ORLICZ-PETTIS-theorem, which was originally proved for weakly sequentially complete BANACH spaces and sequences (ORLICZ [25, p. 244]).

\textbf{(1.2) THEOREM.} For every dual pair \(<E, F>\) the weak topology \( \sigma(E, F) \) and the MACKEY-topology \( \tau(E, F) \) have the same SF-summable families (and therefore also the same BM-summable families).

A connection to the theory of vector valued measures is provided by the following proposition which is easy to prove.

\textbf{(1.3) PROPOSITION.} Let \( E \) be a linear space and \( \mathcal{R} \in \mathcal{T} \) two locally convex topologies on \( E \). Then the following statements are equivalent:

(a) Every sequence in \( E \) which is SF-summable for \( \mathcal{R} \) is also SF-summable for \( \tau \).

(b) If \( \mathcal{A} \) is any \( \sigma \)-ring of sets and \( m: \mathcal{A} \rightarrow E \) is \( \sigma \)-additive for the topology \( \mathcal{R} \) on \( E \), then also \( m: \mathcal{A} \rightarrow (E, \tau) \) is \( \sigma \)-additive.

From (1.2) we obtain in particular that the notions of SF-summability and BM-summability depend only on the dual pair and with (1.3) we see that the same \( E \)-valued set functions are \( \sigma \)-additive for all locally convex topologies on \( E \) which are compatible with \(<E, F>\).

Besides of these immediate applications theorems of the

\[ \text{1) For locally convex spaces (which are tacitly assumed to be separated) and dual pairs we use the notations of HORVÁTH [18].} \]