THEOREMS OF THE ORLICZ-PETTIS-TYPE FOR LOCALLY CONVEX SPACES

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For a dual pair \( \langle E, F \rangle \) we determine the finest \( \langle E, F \rangle \)-polar topology on \( E \) which has the same subfamily-summable (respectively bounded multiplier summable) families as the weak topology \( \sigma(E,F) \). It is shown that these characterizations contain most of the known theorems of the ORLICZ-PETTIS-type for locally convex spaces and also several new results of this type. Some applications to spaces of functions and operators are given.

1. Introduction

Theorems of the ORLICZ-PETTIS-type concern the summability of families with respect to different topologies. We consider the following notions of summability

\[ (\text{a}) (x_L; L \subseteq I) \text{ satisfies the CAUCHY-condition (is summable), if the net } (\sum_{L \subseteq I} x_L; \sigma \in \mathcal{F}(I)) \text{ is a CAUCHY-net (is convergent).} \]

\[ (\text{b}) (x_L; L \subseteq I) \text{ is SF-summable (subfamily-summable) if } (x_L; L \subseteq J) \text{ is summable for all } J \subseteq I. \]

\[ (\text{c}) (x_L; L \subseteq I) \text{ is BM-summable (bounded-multiplier-summable) if } (\lambda_L x_L; L \subseteq I) \text{ is summable for all } \lambda = (\lambda_L; L \subseteq I) \in \mathcal{I}^\omega(I). \]

A family \( (x_L; L \subseteq I) \) satisfies the CAUCHY-condition in \( (E, \mathcal{R}) \) iff one of the following equivalent conditions is satisfied:

\[ (\text{a'}) For every \( \mathcal{O} \)-nbhd. \( U \in \mathcal{V}_0(E, \mathcal{R}) \) there exists \( \sigma \in \mathcal{F}(I) \) with the property: \( \tau \in \mathcal{F}(I), \tau \cap \sigma = \emptyset \implies \]

73
For every \( \mathcal{O} \)-nbhd. \( U \in \mathcal{V}(E, \mathcal{K}) \) there exists \( \mathcal{G} \in \mathcal{F}(I) \) with the property: \( \mathcal{G} \in \mathcal{F}(I), \quad \mathcal{G} \cap \mathcal{O} = \emptyset \Rightarrow \sum_{i \in I} |\langle x_i', x_i \rangle| < 1 \) for all \( x_i' \in U^0 \). \(^1\)

(cf. BOURBAKI[5, p. 64]). From these conditions it is easy to see that \( (x_i; i \in I) \) satisfies the CAUCHY-condition iff every countable subfamily satisfies the CAUCHY-condition.

The following theorem is a locally convex version of the classical ORLICZ-PETTIS-theorem, which was originally proved for weakly sequentially complete BANACH spaces and sequences (ORLICZ[25, p. 244]).

\((1.2)\) THEOREM. For every dual pair \( \langle E, F \rangle \) the weak topology \( \mathcal{G}(E, F) \) and the MACKEY-topology \( \mathcal{T}(E, F) \) have the same SF-summable families (and therefore also the same BM-summable families).

A connection to the theory of vector valued measures is provided by the following proposition which is easy to prove.

\((1.3)\) PROPOSITION. Let \( E \) be a linear space and \( \mathcal{K} \subset \mathcal{T} \) two locally convex topologies on \( E \). Then the following statements are equivalent:

\(a\) Every sequence in \( E \) which is SF-summable for \( \mathcal{K} \) is also SF-summable for \( \mathcal{T} \).

\(b\) If \( \mathcal{M} \) is any \( \mathcal{G} \)-ring of sets and \( m: \mathcal{M} \rightarrow E \) is \( \mathcal{G} \)-additive for the topology \( \mathcal{K} \) on \( E \), then also \( m: \mathcal{M} \rightarrow (E, \mathcal{T}) \) is \( \mathcal{G} \)-additive.

From \((1.2)\) we obtain in particular that the notions of SF-summability and BM-summability depend only on the dual pair and with \((1.3)\) we see that the same \( E \)-valued set functions are \( \mathcal{G} \)-additive for all locally convex topologies on \( E \) which are compatible with \( \langle E, F \rangle \).

Besides of these immediate applications theorems of the

\(^1\)For locally convex spaces (which are tacitly assumed to be separated) and dual pairs we use the notations of HOPVÁTH[18].