ON THE STRUCTURE OF COMMUTATIVE AFFINE GROUP SCHEMES OVER A NON-PERFECT FIELD

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Let $k$ be a non-perfect field of characteristic $p > 0$ with a $p$-basis $\beta$ and $k_s$ the algebraic separable closure of $k$. Starting from the ring of Schoeller $D^\beta$ [3] and the topological Galois group $\Pi$ of $k_s$ over $k$, we construct a new ring $\phi$ such that the category of commutative affine $k$-group schemes is anti-equivalent to the category of effaceable left $\phi$-modules. (The effaceability is defined in the text).

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Introduction

Let $k$ be a non-perfect field of characteristic $p > 0$ with a $p$-basis $\beta$. Let $k_s$ be the algebraic separable closure of $k$ and $\Pi$ the topological Galois group of $k_s$ over $k$. A left $\Pi$-module is continuous if the stabilizer of its each element is an open subgroup of $\Pi$. Then the category of commutative affine multiplicative $k$-group schemes is anti-equivalent to the category of continuous left $\Pi$-modules [1,IV,§1,3.6]. On the other hand C. Schoeller [3] describes a ring $D^\beta$ and a family of left
ideals of \( D^\infty \{L_\alpha \} \), calls a left \( D^\infty \)-module effaceable if the annihilator of its each element contains some of \( L_\alpha \) and proves that \( \text{Acu}_k \), the category of commutative affine unipotent \( k \)-group schemes is anti-equivalent to the category of effaceable left \( D^\infty \)-modules. We recall the definition of the ring \( D^\infty \) and the family \( \{L_\alpha \} \) in \$1, correcting some errors of the original description (mainly in case \( \text{card}(\mathcal{B}) \) is infinite).

Let \( \text{Acu}_k \) denote the category of commutative affine \( k \)-group schemes. The purpose of this paper is to construct a ring \( \phi \) and a family of its left ideals \( \{L_\beta \} \) in such a way that the category \( \text{Acu}_k \) is anti-equivalent to the category of left effaceable \( \phi \)-modules, where we call a left \( \phi \)-module effaceable if the annihilator of its each element contains some of the left ideals \( L_\beta \). The ring \( \phi \) which we describe in this paper has the following form: We determine a \( D^\infty \)-\( \mathbb{H} \)-bimodule \( \mathcal{E} \) and a family of its sub-bimodules \( \{E_\gamma \} \) in some canonical way. The quotient bimodules \( \mathcal{E}/E_\gamma \) are finitely generated effaceable left \( D^\infty \)-modules and right continuous \( \mathbb{H} \)-modules. Let \( \phi = \begin{bmatrix} D^\infty & \mathcal{E} \\ 0 & \mathbb{Z}[\mathbb{H}] \end{bmatrix} \) be

the set of \( 2 \times 2 \) matrices \( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \) with \( a \in D^\infty, b \in \mathcal{E} \) and \( c \in \mathbb{Z}[\mathbb{H}] \) (= the group ring of \( \mathbb{H} \)). The usual matrix multiplication rule makes \( \phi \) a ring. (Notice that \( D^\infty \mathcal{E} + \mathcal{E} \mathbb{H} \subseteq \mathcal{E} \). The families \( \{L_\alpha \} \) and \( \{E_\gamma \} \) and the topology of \( \mathbb{H} \) determine a family of left ideals of \( \phi \) \( \{L_\beta \} \) in some canonical way. The ring \( \phi \) and the family \( \{L_\beta \} \) satisfy the condition we mentioned above. The main theorem on the structure of \( \text{Acu}_k \) is proved in \$2. A more close observation of the bimodule \( \mathcal{E} \) is made in \$3.

Our notation is essentially the same as [1] and [3]. If \( \mathcal{C} \) is a category, \( a \in \mathcal{C} \) means that \( a \) is an object of \( \mathcal{C} \). If \( a, b \in \mathcal{C}, \mathcal{C}(a,b) \) denotes the class of morphisms from \( a \) to \( b \). \( \mathcal{M} \) and \( \mathcal{E} \) denote the categories of models and sets respectively. A model means a small commutative ring. If \( K \in \mathcal{M}, \mathcal{M}_K \) denotes the category of \( K \)-models (i.e., small commutative \( K \)-algebras) and \( \mathcal{M}_K \mathcal{E} \) the category