New perspective on Poisson’s ratios of elastic solids

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Summary. A new perspective on Poisson’s ratios of elastic solids is presented. We show that, by scaling the Poisson’s ratios through the square root of a modulus ratio, the transformed Poisson’s ratios, $n_1$, $n_2$, $n_3$, are bounded in a closed region, which is inside a cube centered at the origin with a range from $-1$ to $1$. The shape of this closed region, depicted in Fig. 1, looks like a Chinese food, “Zongzi”. With this geometric interpretation, any positive definite compliance of an orthotropic solid can be easily constructed by selecting any point inside the region, together with any three positive Young’s moduli and any three positive shear moduli. This provides a new insight to the admissible range of Poisson’s ratios. We also provide an example that the inequality proven by Rabinovich [6], i.e. $\nu_{12} + \nu_{23} + \nu_{31} \leq 3/2$, is not generally true.

1 Introduction

Admissible ranges of elastic properties are of fundamental interest. In this paper we present a new interpretation regarding the Poisson’s ratios of anisotropic solids. Poisson’s ratios have long been an interesting research subject (see, for example, Rothenburg et al. [7], Milton [5], Lakes [2], [3] and the references cited therein). The present work is to explore geometrically what the admissible ranges of three Poisson’s ratios in any three mutually orthogonal planes are.

By a change of scales in the Poisson’s ratios through the ratios of Young’s modulus, we show that the transformed Poisson’s ratios are bounded in a closed body, which is inside a cube centered at the origin with a range from $-1$ to $1$. Its geometry (Fig. 1) resembles a kind of traditional Chinese food “Zongzi”. “Zongzi” is a pyramid-shaped dumpling made of glutinous rice wrapped in bamboo or reed leaves eaten during the Dragon Boat Festival, having a history more than two millenniums in memory of the great poet and politician Qi Yuan (BC 340–278). With this geometric interpretation, any positive definite compliance of an orthotropic solid can be easily constructed by selecting any point inside the region, together with any three positive Young’s moduli and any three positive shear moduli. This provides a clear image on the admissible range of Poisson’s ratios. Evidently, by selecting sufficiently large or small Young’s moduli, it is readily seen that Poisson’s ratios of elastic compliance need not be either bounded above or bounded below, a previously known result.

In addition, we also provide a counter-example that the inequality of Poisson’s ratios (Lekhnitskii [4], Rabinovich [6]) referred to an arbitrary Cartesian system,

$$ R = \nu_{12} + \nu_{23} + \nu_{31} \leq 3/2, $$

is not generally true.
2 Positive definiteness of an orthotropic compliance

Let us first consider an orthotropic solid and assign a Cartesian coordinate system \( \{x_i\} \) with respect to the three mutually orthogonal privileged directions of the orthotropic solid. Thus, the Voigt matrix of the elastic compliance can be written in the form

\[
S = \begin{bmatrix}
\frac{1}{E_1} & -\nu_{12}/E_1 & -\nu_{13}/E_1 & 0 & 0 & 0 \\
-\nu_{21}/E_2 & \frac{1}{E_2} & -\nu_{23}/E_2 & 0 & 0 & 0 \\
-\nu_{31}/E_3 & -\nu_{32}/E_3 & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \\
\end{bmatrix},
\]

where \( E_i \) is Young’s modulus in the \( x_i \) direction; \( G_{ij} \) is the shear modulus for planes parallel to the \( x_i-x_j \) plane; the Poisson’s ratio \( \nu_{ij} \) characterizes the contraction in the direction of the \( x_j \) axis when a uniaxial stressing is applied in the \( x_i \) direction. Diagonal symmetry of the compliance matrix ensures that

\[
\nu_{23}/E_2 = \nu_{32}/E_3, \quad \nu_{31}/E_3 = \nu_{13}/E_1, \quad \nu_{12}/E_1 = \nu_{21}/E_2.
\]

The thermodynamic constraints on the values of the elastic constant require that the compliance matrix must be positive definite. This implies that any principal minor of \( S \) is positive. For example, the diagonal entries

\[
E_1 > 0, \quad E_2 > 0, \quad E_3 > 0, \quad G_{23} > 0, \quad G_{31} > 0, \quad G_{12} > 0
\]

are all positive. To find the remaining constraints, we introduce the positive definite matrix

\[
J = \text{diag} \left( \sqrt{E_1}, \sqrt{E_2}, \sqrt{E_3}, \sqrt{G_{23}}, \sqrt{G_{31}}, \sqrt{G_{12}} \right).
\]

Back to (2), we can equivalently consider \( S' = JSJ \),

\[
S' = \begin{bmatrix}
1 & -n_3 & -n_2 & 0 & 0 & 0 \\
-n_3 & 1 & -n_1 & 0 & 0 & 0 \\
-n_2 & -n_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]