Hencky's elasticity model and linear stress-strain relations in isotropic finite hyperelasticity

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Dedicated to Prof. Dr.-Ing. Otto Timme Bruhns on the occasion of his 60th birthday

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Summary. Hencky's elasticity model is an isotropic finite elasticity model assuming a linear relation between the Kirchhoff stress tensor and the Hencky or logarithmic strain tensor. It is a direct generalization of the classical Hooke's law for isotropic infinitesimal elasticity by replacing the Cauchy stress tensor and the infinitesimal strain tensor with the foregoing stress and strain tensors. A simple, straightforward proof is presented to show that Hencky's elasticity model is exactly a hyperelasticity model, derivable from a quadratic potential function of the Hencky strain tensor. Generally, Hill's isotropic linear hyperelastic relation between any given Doyle-Ericksen or Seth-Hill strain tensor and its work-conjugate stress tensor is studied. A straightforward, explicit expression of this general relation is derived in terms of the Kirchhoff stress and left Cauchy-Green strain tensors. Certain remarkable properties of Hencky's model are indicated from both theoretical and experimental points of view.

1 Introduction

The stress-strain relation for an isotropic elastic solid at infinitesimal deformations is characterized by the commonly known Hooke's law

\[ \sigma = \Lambda(\varepsilon) \mathbf{I} + 2G \varepsilon, \]

i.e.,

\[ \sigma = \mathbf{S} : \varepsilon. \]

In the above, \( \sigma \) is the Cauchy stress tensor, \( \varepsilon \) the infinitesimal strain tensor, \( \Lambda \) and \( G \) the Lamé elastic constants, and \( \mathbf{S} \) the classical isotropic elastic stiffness tensor. Throughout, \( \mathbf{I} \) is used to designate the second-order identity tensor and \( \text{tr} \mathbf{A} \) the trace of the second-order tensor \( \mathbf{A} \), i.e., \( \text{tr} \mathbf{A} = A_{ii} \).

In general, with reference to an undistorted referential configuration the stress-strain relation for an isotropic elastic solid at arbitrary infinitesimal or finite deformations may be described by a general form of nonlinear isotropic tensor-valued stress response function \( \Phi(B) \) of the Cauchy-Green tensor \( B = FF^T \), i.e.,

\[ \sigma = \Phi(B). \]

An elastic relation described above is in Cauchy's sense and defined by a tensor-valued stress response function \( \Phi(B) \). It is said to be hyperelastic or in Green's sense, whenever there is an isotropic potential function

\[ \Sigma = \Sigma(B), \]
called strain-energy function, such that the material time derivative \( \dot{\Sigma} \) supplies the specific stress power per unit reference volume, i.e.,

\[
\dot{\Sigma} = \text{tr} (\tau \mathbf{D}).
\]

(5)

Throughout, \( \tau \) and \( \mathbf{D} \) are used to represent the Kirchhoff stress tensor and the Eulerian strain-rate or stretching tensor. The former is simply the Cauchy stress multiplied by the volumetric ratio \( J \), i.e.,

\[
\tau = J \sigma, \quad J = \det \mathbf{F}.
\]

(6)

In the above, \( \det \mathbf{F} \) stands for the determinant of the deformation gradient \( \mathbf{F} \). Owing to the condition (5), the stress-strain relation for an isotropic hyperelastic solid is simplified and derivable from the strain-energy function \( \Sigma \), refer to e.g. Truesdell and Noll [1] and Ogden [2] for details.

Hooke’s law (1) is an isotropic hyperelastic relation for infinitesimal deformations, characterized by the two Lamé constants \( \Lambda \) and \( G \). For them, there have been well-documented experimental data available for various kinds of engineering materials. However, if the deformation is beyond the range of infinitesimal deformations, even it might not be easy to determine the form of the strain-energy function \( \dot{\Sigma}(\mathbf{B}) \) for a realistic elastic material of interest by fitting experimental data, let alone determine the tensor-valued stress response function \( \phi(\mathbf{B}) \). For the sake of simplicity and for purposes of computations and practical applications, it appears to be attractive and desirable to find out a direct finite strain generalization of the classical Hooke’s law (1) that is characterized simply by the two classical Lamé elastic constants \( \Lambda \) and \( G \) evaluated at infinitesimal deformations, and, at the same time, can reasonably approximate elastic behavior of realistic engineering materials within a suitable range of finite deformations. This may be done by replacing \( \sigma \) and \( \epsilon \) in (1) or (2) with a stress tensor \( \tau \) and a finite strain tensor \( \mathbf{E} \). Then, we have the generalization

\[
\tau = \Lambda (\text{tr} \mathbf{E}) \mathbf{I} + 2 G \mathbf{E}.
\]

(7)

In general, the above relation need not be hyperelastic. It is hyperelastic only when the stress-strain pair \((\tau, \mathbf{E})\) therein is a work-conjugate stress-strain pair fulfilling the condition (24) given later.

In the past decades, examples of the above linear stress-strain relation for isotropic finite elasticity have been proposed and used for incompressible and compressible cases. For instance, we have the St. Venant-Kirchhoff’s model (see [1]) with \( \tau = \mathbf{F}^{-1} \sigma \mathbf{F}^{-T} \) and \( \mathbf{E} = \frac{1}{2} (\mathbf{B} - \mathbf{I}) \), the Signorini’s model (see [1]) with \( \tau = \sigma \) and \( \mathbf{E} = \frac{1}{2} (\mathbf{I} - \mathbf{B}^{-1}) \), the Hencky’s model (see Hencky [3]–[5]) with \( \tau = \mathbf{e} \) and \( \mathbf{E} = \frac{1}{2} \ln \mathbf{B} \), and the incompressible neo-Hookean model (see Rivlin [6]) with \( \tau = \sigma \) and \( \mathbf{E} = \frac{1}{2} (\mathbf{B} - \mathbf{I}) \) and with the first term in Eq. (7) replaced by \(-\rho \mathbf{I}\), etc. A variety of examples were studied by Seth [7]. A general case for linear hyperelastic stress-strain relations was proposed by Hill [8].

In a very recent paper, Chiskis and Parnes [9] have considered the following interesting question: For what strain measure \( \mathbf{E} \) is Eq. (7) with \( \tau = \sigma \), i.e., the equation

\[
\sigma = \Lambda (\text{tr} \mathbf{E}) \mathbf{I} + 2 G \mathbf{E}
\]

(8)

hyperelastic? They have found that the strain measure \( \mathbf{E} \) should assume the form (see Eq. (3.9) in [9])

\[
\mathbf{E} = \frac{\Lambda}{2 G} (\mathbf{B}^{\Lambda/\Lambda} - \mathbf{I}).
\]

(9)

The shear modulus \( \mu \) and the left stretch tensor \( \mathbf{V} \) therein are replaced by \( G \) and \( \mathbf{B} \) here.