Convergence of Stationary Sequences for Variational Inequalities with Maximal Monotone Operators

A. Auslender

Department of Applied Mathematics, University Blaise Pascal, 63170 Aubière Cedex, France

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Abstract. Let $T$ be a maximal monotone operator defined on $\mathbb{R}^N$. In this paper we consider the associated variational inequality $0 \in T(x^*)$ and stationary sequences $\{x_k^*\}$ for this operator, i.e., satisfying $T(x_k^*) \to 0$. The aim of this paper is to give sufficient conditions ensuring that these sequences converge to the solution set $T^{-1}(0)$ especially when they are unbounded. For this we generalize and improve the directionally local boundedness theorem of Rockafellar to maximal monotone operators $T$ defined on $\mathbb{R}^N$.

Key Words. Maximal monotone operators, Convex programming, Variational inequalities.

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1. Introduction

In what follows we assume a certain familiarity with convex analysis for which we basically follow Rockafellar's book. We denote $X = \mathbb{R}^N$, $(\cdot, \cdot)$ and $\|\cdot\|$ are the usual inner product and the associated norm on $X$, and $d(x, C)$ is the usual Euclidean distance to the set $C$.

A monotone operator $A$ from $X$ to $X$ (generally multivalued) is a mapping such that

$$(x - y, x^* - y^*) \geq 0 \quad \text{whenever } x^* \in A(x), \ y^* \in A(y). \quad (1.1)$$

Its domain denoted $\text{dom} \ A$ is the set of $x$ such that $A(x)$ is nonempty. Such an operator is said to be maximal if its graph is not properly contained in the graph
of any other monotone operator $A'$ from $X$ to $X$. In other words, $A$ is maximal iff, for each $x, x^*$ satisfying

$$ (y^* - x^*, y - x) \geq 0, \quad \forall y^* \in A(y), \quad \forall y \in \text{dom } A, $$

$$ x^* \in A(x). $$

Subdifferentials of convex functions are monotone maximal operators from $X$ to $X$ and monotone maximal operators enjoy many properties of subdifferentials of convex functions. Maximal monotone operators have been studied extensively because of their role in convex analysis, in certain partial differential equations, and in traffic equilibrium problems.

Let $T = A^{-1}$ be the inverse operator of $A$, i.e., $A^{-1}(x^*) = \{x : x^* \in A(x)\}$. $T$ is also a maximal monotone operator and let us consider the variational inclusion

$$ \text{(P)} \quad 0 \in T(x^*) $$

for which we denote the solution set by $S = A(0)$.

This paper addresses a question that arises in the study of algorithms for solving (P). Such algorithms typically generate a sequence $\{x^*_n\}$ in $X$ such that $d(0, T(x^*_n)) \to 0$. The performance of such methods can be regarded as satisfactory if $d(x^*_n, S) \to 0$. However, we have only a few results that assert this property.

Under the assumption

$$ 0 \in \text{int } \text{dom } A \quad \text{(1.4)} $$

it is well known that (1.3) has a nonempty compact set of solutions $S = A(0)$. Furthermore, if $\{x^*_n\}$ is a sequence such that $d(0, T(x^*_n)) \to 0$, then this sequence is bounded and $d(x^*_n, S) \to 0$. This property is not true in general when (1.4) is no longer assumed. Indeed, when $T$ is the subdifferential of a closed proper convex function defined on $X$, then (P) becomes the extremum problem

$$ m = \inf(f(x) | x \in X) $$

and if we consider the function given by Rockafellar [5],

$$ f(x_1, x_2) = \begin{cases} \frac{x_2^2}{x_1} & \text{if } x_1 > 0, \\ 0 & \text{if } x_1 = x_2 = 0, \\ +\infty & \text{elsewhere,} \end{cases} $$

then

$$ m = 0, \quad \nabla f(p_n^{3/2}, p_n) = \left( \frac{-1}{p_n}, \frac{2}{\sqrt{p_n}} \right), \quad f(p_n^{3/2}, p_n) = \sqrt{p_n}, $$

and

$$ \lim_{p_n \to +\infty} \nabla f(p_n^{3/2}, p_n) = 0, \quad \lim_{p_n \to +\infty} f(p_n^{3/2}, p_n) = +\infty. $$

In this case the sequence $\{p_n^{3/2}, p_n\}$ is unbounded, $d((p_n^{3/2}, p_n), S) \not\to 0$, and we can