Complex hypersingular integrals and integral equations in plane elasticity

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(Received January 11, 1992; revised March 8, 1993)

Summary. Complex hypersingular (finite-part) integrals and integral equations are considered in the functional class of N. Muskhelishvili. The appropriate definition is given. Three regularization (equivalence) formulae follow from this definition. They reduce hypersingular integrals to singular ones and allow to derive hypersingular analogues for Sokhotsky-Plemelj's formulae and for conditions that are necessary and sufficient for the function to be piecewise holomorphic. Two approaches to get and investigate complex hypersingular equations follow from these results: one of them is based on the equivalence formulae; as to the other, it is based on above-mentioned conditions. As an example, authors' equation for plane elasticity is studied. The existence of a unique solution is stated and some advantages over singular equations are outlined. To solve hypersingular equations the quadrature rules are presented. The accuracy of different quadrature formulae is compared, the examples being used. They confirm the need to take into account asymptotics and to carry out a thorough analytical investigation to get safe numerical results.

1 Introduction

Recent investigations in fracture mechanics and elasticity increased the interest in hypersingular integral equations [1]—[14]. Integrals in these equations do not exist as ordinary, improper or even Cauchy principal value integrals. Their meaning was shown by Hadamard (see [15]) who was the first to define them in terms of the distributions theory. These integrals proved to be useful in various fields: aerodynamics [16], [17], [19], acoustic and scatter problems [13], [18], [20], [21], plane and three-dimensional elasticity [1]—[14]. It is also worth to note that the popular displacement discontinuity method [22] is, in essence, the simplest (piecewise constant) numerical realization of the appropriate hypersingular equation [12].

In this paper complex hypersingular integrals and integral equations are considered. To the best of our knowledge hypersingular equations until now have been employed only for real functions and variables. But complex variables have their significant virtues that impel to use them. Using them, one can apply highly developed theories of analytical functions, approximations, singular integrals and equations [23]. Computational advantages are also clear. Since modern computers handle complex arithmetic, we can gain when using in plane problems one (complex) function of one (complex) variable.

A complex hypersingular equation was suggested by the authors in [12] for an elastic plane with cracks. Below we present some results that can provide theoretical investigations of such equations and their numerical solution.
2 Complex finite-part integrals

We assume that the contour $L$ is a piecewise smooth curve in a plane $z = x + iy$. It consists of $n$ smooth open arcs $L_k$ with continuous curvature. The beginning of the $k$-th arc we denote as $a_k$, its end being $b_k$. Some of this points of $L$ may coincide. These coinciding points and edges of $L$ will be called nodes. We denote them as $C_k$. The other points of $L$ we shall call ordinary points.

Assume also that for any arc $L_k (k = 1, \ldots, n)$ a function $g_k(t)$ is given. Its derivative $g'_k(t)$ belongs to $H^*$ class of N. Muskhelishvili [23]. The set of functions $g_k(t) (k = 1, \ldots, n)$ defines a function $g(t)$ of $H_0$ class on $L$. This function is defined in ordinary points of $L$; its derivative $g'(t)$ is a function of $H^*$ class on $L$.

We shall consider the integral

$$
\Phi(z) = \frac{1}{2\pi i} \int_L \frac{g(t) \, dt}{(t - z)^2}.
$$

(2.1)

It can be called Hadamard type integral.

$\Phi(z)$ is a holomorphic function in any region that does not contain points of $L$. Now we intend to consider such points. To do this we assume for a while that the contour $L$ consists of only one open smooth arc $(ab)$ (Fig. 1). The general case will be naturally included below.

Embrace any ordinary point $t_0$ with a circle of sufficiently small radius $\varepsilon$ to cross $L$ in two ordinary points $t_1$ and $t_2$. Excluding the small arc $l = t_1t_2$ from integration we have

$$
\Phi_v(t_0) = \frac{1}{2\pi i} \int_{L-l} \frac{g(t) \, dt}{(t - t_0)^2}.
$$

This integral exists as an ordinary integral. It can be written in the following form:

$$
\Phi_v(t_0) = \frac{1}{2\pi i} \int_{L-l} \frac{g(t) - g(t_0)}{(t - t_0)^2} \, dt + g(t_0) \int_{L-l} \frac{dt}{(t - t_0)^2}.
$$

(2.2)

Fig. 1. Geometry and co-ordinate system