COHEN-MACAULAY PROPERTIES OF THE KOSZUL HOMOLOGY

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The Koszul homology $H.(y,N)$ which is constructed with respect to a sequence $y$ and a maximal Cohen-Macaulay (CM) module $N$ over a local CM ring $A$ admitting a canonical module $\omega_A$, will be compared with the Koszul homology $H.(y,\text{Hom}_A(N,\omega_A))$.

If $R := A/I$ with $I = (y)$ is a CM ring, then the canonical module $\omega_R$ of $R$ exists and we will mainly show the existence of a natural isomorphism $H.(y,\text{Hom}_A(N,\omega_A)) \simeq \text{Hom}_R(H.(y,N),\omega_R)$, if $H.(y,N)$ is a maximal CM module over $R$. This generalizes a result of Herzog in [2].

Using this isomorphism we are able to compute the graded canonical module of the graded ring $\text{gr}_I(A)$ in a certain case.

In the last part of this paper we define a polynomial $U^N(y,x)$ associated with the Koszul homology $H.(y,N)$ similar to Huneke in [7]. Huneke proved that $H^j(y,N)$ is CM, if $j < \text{mindeg} \ U^N(y,x)$. We will proceed to show that $H^j(y,N)$ is CM, if $j > \text{deg} \ U^N(y,x)$.

INTRODUCTION

Throughout this paper we will be dealing with commutative rings with identity, which are noetherian.

Let $A$ be such a ring. Then we only consider finitely generated modules over $A$.

If in addition $A$ is local with maximal ideal $m$ and residue class field $k$ and if $M$ and $N$ are two modules over $A$, we

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1 The material presented in this paper constitutes part of the author's thesis submitted to Universität Essen.
define \( \text{grade}^N_M := \inf \{ i | \text{Ext}_A^i(M, N) \neq 0 \} \). Notice that \( \text{grade}^N_M \) is the maximal length of an \( N \)-regular sequence in \( \text{Ann} M \).

For an ideal \( I \) in \( A \) it is convenient to put \( \text{grade}^A_I R \), where \( R = A/I \).

Further if \( A \) is a Cohen-Macaulay (CM) ring admitting a canonical module \( \omega_A \), we define the dual module of \( N \) over \( A \) as \( \text{Ext}_A^{d-e}(N, \omega_A) \) where \( e := \dim N \) and \( d = \dim A \) and denote it by \( \omega_A(N) \).

If \( N \) is CM of dimension \( e \), then \( \omega_A(N) \) is also CM of dimension \( e \) and \( \omega_A(\omega_A(N)) = N \). In particular \( \omega_A(\omega_A) = A \).

For more details see [3].

We now make some assumptions which are valid for this paper.

Let \((A, m, k)\) be a \( d \)-dimensional local CM ring admitting a canonical module \( \omega_A \). Further let \( I \) be an ideal in \( A \) with \( \text{grade} I = g \) which is generated by the sequence \( y = y_1, \ldots, y_n \). Put \( R = A/I \).

We think of the Koszul complex \( K_\bullet(y, A) \) as follows. \( K_1(y, A) \) is a free \( A \)-module of rank \( n \) with basis \( e_1, \ldots, e_n \). For each \( p = 0, \ldots, n \) let \( K_p(\cdot, A) := \Lambda^p K_1(y, A) \). We define the boundary map \( \partial : K_p(y, A) \rightarrow K_{p-1}(y, A) \) by its action on the basis vectors:

\[
\partial(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} y_{i_j} e_{i_1} \wedge \ldots \wedge \hat{e}_{i_j} \wedge \ldots \wedge e_{i_p}
\]

(here \( \wedge \) means omitting that element). Thus \( K_\bullet(y, A) \) is a (homological) complex of \( A \)-modules. If \( N \) is any \( A \)-module we set \( K_\bullet(y, N) := K_\bullet(y, A) \otimes_A N \). Its cycles, respectively its homology, we will denote by \( Z_\bullet(y, N) \), respectively by \( H_\bullet(y, N) \).