Automorphism groups of algebraic lattices

GERHARD BEHRENDT

Abstract. We classify those algebraic lattices whose group of automorphisms is transitive on the set of elements of the lattice except the smallest and the greatest. We describe their automorphism groups in terms of generalized wreath powers.

0. Introduction

There is a multitude of algebraic lattices whose automorphism group has as fixed elements only the smallest and the greatest element of the lattice (e.g. the normal subgroup lattices of characteristically simple groups). If, however, we consider algebraic lattices with the stronger property that the automorphism group acts transitively on the lattice except the smallest and the greatest element, then we can classify those completely.

For the standard concepts we refer the reader to Grätzer [3]. In the following we shall need the fact that any algebraic lattice is join-continuous ([2], 2.3), and that in an algebraic lattice an element is compact if and only if it is join-inaccessible ([1], VIII.5, Lemma 3).

If \( \Omega \) is a set with an element \( 0 \in \Omega \) then we define \( \Omega^{(0)} := \{ \omega \in \Omega^N \mid \omega_i = 0 \text{ for all but finitely many } i \} \). We further define \( CT(\Omega) := \mathbb{Z} \times \Omega^{(0)} \), and we partially order \( CT(\Omega) \) by \( (z, \omega) \leq (z', \omega') \) if and only if \( z \leq z' \) and \( \omega_i' = \omega_i + z' - z \). If \( L = CT(\Omega) \cup \{0; 1\} \) where \( 0 < c < 1 \) for all \( c \in CT(\Omega) \) then it is easy to see that \( L \) is an algebraic lattice.

1. The classification of the lattices

We want to classify those algebraic lattices \( L \) whose automorphism group is transitive on \( L^* = L \setminus \{0; 1\} \).

Presented by Bjarni Jónsson. Received September 26, 1983. Accepted for publication in final form May 10, 1985.
LEMMA 1. Let $L$ be an algebraic lattice such that each element of $L^* = L \setminus \{0; 1\}$ is join-inaccessible, and suppose $L^* \neq \emptyset$. Then there exists an element $t$ of $L^*$ which is covered by an element $u$ such that for $s \in L$ if $t < s$ then $u \leq s$.

Proof. We can assume that there exist two incomparable elements $b, c$ whose join $a_0$ lies in $L^*$. Then $a_0$ covers two distinct elements $b_0, c_0$ (which can be chosen as maximal among the elements smaller than $a_0$ but greater than $b$ or $c$ respectively). Now assume the lemma to be false. Then there exists an element $d_0 \in L$ such that $c_0 < d_0$ but $a_0 \neq d_0$. Now suppose we have constructed $a_0, b_0, c_0, d_0$ such that $a_i$ covers $b_i$ and $c_i$, and such that $d_i \in L^*$ with $c_i < d_i$ but $a_i \neq d_i$. Then $a_{i+1} := a_i \lor d_i$. Choose $b_{i+1} \leq a_i$ and $c_{i+1} \geq d_i$, both covered by $a_{i+1}$. Note that $b_{i+1} \neq c_{i+1}$.

Let $d_{i+1}$ be an element of $L^*$ with $c_{i+1} < d_{i+1}$ and $a_{i+1} \neq d_{i+1}$.

Now $c_0 \leq a_0 \land c_i \leq a_i$, hence we have either $a_0 \land c_i = c_0$ or $a_0 \land c_i = a_0$ for $i \in \mathbb{N}$. Suppose there exists $n \in \mathbb{N}$ such that $a_0 \land c_n = a_0$, hence $a_0 \leq c_n$. As $d_i \leq c_n$ for $i < n$ we get $a_n = a_0 \lor d_{n+1} \leq c_n$, and so forth.

Finally, we have $a_n = a_{n-1} \lor d_{n-1} \leq c_n$, which is clearly a contradiction. Therefore we must have $a_0 \land c_i = c_0$ for all $i \in \mathbb{N}$. As $c_i < c_{i+1}$ for $i \in \mathbb{N}$ and $\lor \{c_i \mid i \in \mathbb{N}\} = 1$, and hence, as $L$ is join-continuous, we have $a_0 = a_0 \lor \lor \{c_i \mid i \in \mathbb{N}\} = \lor \{a_0 \land c_i \mid i \in \mathbb{N}\} = c_0$, which is a contradiction. This proves the lemma.

THEOREM 2. Let $L$ be an algebraic lattice such that the automorphism group of $L$ is transitive on $L^* = L \setminus \{0; 1\}$. If $L^* \neq \emptyset$ then either all elements of $L^*$ are pairwise incomparable or there exists a non-empty set $\Omega$ such that $L^*$ is order-isomorphic to $CT(\Omega)$.

Proof. First note, that as there must exist join-inaccessible elements in $L^*$, all elements of $L^*$ must be join-inaccessible by transitivity. Then, by Lemma 1 and transitivity, each element $r$ of $L^*$ is covered by an element $u(r)$ such that for $x \in L$ if $r < x$ then $u(r) \leq x$. If one element of $L^*$ is covered by 1 then all elements of $L^*$ are pairwise incomparable. So let us suppose that no element of $L^*$ is covered by 1. It is easy to see that for any $r \in L^*$ the set $\{x \in L^* \mid x \geq r\}$ is order-isomorphic to the set of natural numbers, and hence we can write $\{x \in L^* \mid x \geq r\} = \{n_i \mid i \in \mathbb{N}\}$ with $n_i < n_{i+1}$.

Next note that for $a, b \in L^*$ we have $a \lor b \neq 1$. For, if $a, b \in L^*$ then $\lor \{a_i \mid i \in \mathbb{N}\} = 1$, and, as $L$ is join-continuous, $b = b \lor \lor \{a_i \mid i \in \mathbb{N}\} = \lor \{b \land a_i \mid i \in \mathbb{N}\}$. As $b$ is join-inaccessible, there exists $j \in \mathbb{N}$ such that $b \land a_j = b$, hence $b \leq a_j$. Note that if $j$ is chosen minimal with this property then $a \lor b = a_j$.

Now for $a, b \in L^*$ we define $a \sim b$ if and only if there exists $j \in \mathbb{N}$ such that $a_j = b_j$. Note that this gives an equivalence relation on $L^*$.

Now let $\{a_z \mid z \in \mathbb{Z}\}$ be a maximal chain in $L^*$ (where $a_{z+1}$ covers $a_z$). Then