Iterative Minimization of Quadratic Functionals*

Irwin W. Sandberg

Abstract. A complete solution is given to the problem of determining when a certain iteration will converge to the minimum of an important type of quadratic functional. It is shown that convergence occurs whenever the minimum exists, and that the iterates produced by the iteration will be unbounded for every starting point if the minimum does not exist. Applications are given concerning an adaptive filtering algorithm and nonharmonic series expansions.

1. Introduction

Problems involving the minimization of quadratic functionals arise in several engineering areas (see, for example, [6], [8], [12]). For instance, in the adaptive filtering area [6], [12], and for reasons having to do with certain stochastic models, interest centers around a successive-approximation approach to the minimization with respect to $x$ of an expression of the form

$$J(x) = x^T R x - 2s^T x$$

in which $x$ belongs to the set $\mathbb{R}^m$ of real column $m$-vectors (where $m$ is an arbitrary positive integer), $R$ is a nonnegative definite real symmetric matrix, $s \in \mathbb{R}^m$, and the superscript $T$ denotes transpose. Typically it is assumed that $R$ is positive definite, in which case (1) is minimized at the unique point in $\mathbb{R}^m$ that satisfies $Rz = s$, and $z$ is the limit of the iterates $x_1, x_2, \ldots$ defined by

$$x_{n+1} = x_n - c R x_n + c s$$

* Received June 5, 1993; revised September 2, 1993.

1 Department of Electrical and Computer Engineering, The University of Texas at Austin, Austin, Texas 78712.

2 Quadratic minimization problems in the electrical engineering area date back to the work of Wiener.
for sufficiently small positive $c$ and any starting point $x_0$. Thus, for small enough $c$, $J(x_n)$ converges to the minimum of (1) as $n \to \infty$.

In this paper we consider the minimization problem and (2) in the general setting in which $R$ is not necessarily invertible. We show that for any starting point $x_0$ the iteration (2) generates a convergent sequence that minimizes $J(x)$ whenever $J(x)$ is bounded from below. More importantly, we also give corresponding results for the case in which (1) is replaced with

$$
\langle Qx, x \rangle - 2\langle p, x \rangle, \quad x \in H
$$

where $H$ is a real Hilbert space, $Q$ is a linear self-adjoint nonnegative map of $H$ to $H$, and $p \in H$. This allows consideration of cases in which, for example, $x$ is a square integrable function.

In Section 2 we motivate consideration of (3) by formulating a deterministic Hilbert space version of a familiar problem. Our main result is the theorem in Section 3. It concerns (3) and the iteration

$$
x_{n+1} = (I - cQ)x_n + cp, \quad n \geq 0
$$

where $I$ is the identity map on $H$, $c$ is a positive constant bounded in a certain way, and $x_0 \in H$ is arbitrary. According to the theorem, $J(x_n)$ approaches $\min_x J(x)$ whenever this minimum exists, $\min_x J(x)$ exists if and only if there is a solution $x$ of $Qx = p$, and if $\min_x J(x)$ does not exist and the set $Q(H)$ is a closed subset of $H$ (which is the case if $H$ is finite dimensional), then $\min_x J(x) = -\infty$.

As a simple application of our results, we note that they establish the convergence of the expected value of the weight vector in the LMS algorithm (see [12, pp. 101-102]) under much weaker assumptions than typically assumed [i.e., without the invertibility of the covariance matrix (here $R$ plays the role of this matrix)].

In particular, they establish universal convergence for sufficiently small values of the adaptation parameter under the usual statistical independence assumptions. An application concerning nonharmonic series expansions is given in Appendix 4.

In this connection, the approach to minimizing (1) in the adaptive filtering area is to employ a gradient search technique. One could approach the problem differently by recalling the contraction mapping fixed-point theorem and observing that $f$ defined by $f(x) = x - cRx + cs$ is a contraction mapping of $\mathbb{R}^m$ into $\mathbb{R}^m$ for sufficiently small $c$. In fact, the LMS algorithm could have been developed without any reference to a gradient search. This is consistent with the fact that the proof of our theorem relies on certain results concerning nonexpansive maps.

---

3 An early signal-theory paper in which it is shown that equations of the form $Rz = s$ can be solved using this iteration is [11].

4 See the corollary in Section 3.1. In the case of the LMS algorithm the associated quadratic form (with the constant term) is nonnegative and hence bounded from below. Thus, for this case, the solution set $S$ of Section 3 is not empty.

5 This simple result (see Section 3.1) is probably known, but the writer has not found it in the literature.