On the Parameterization of Algebraic Curves*

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Received November 12, 1990; revised version March 24, 1992

Abstract. In this paper, by using the concept of resolvents of a prime ideal introduced by Ritt, we give methods for constructing a hypersurface which is birational to a given irreducible variety and birational transformations between the hypersurface and the variety. In the case of algebraic curves, this implies that for an irreducible algebraic curve $C$, we can construct a plane curve which is birational to $C$. We also present a method to find rational parametric equations for a plane curve if it exists. Hence we have a complete method of parameterization for rational algebraic curves.

Keywords: Geometric modeling, Parameterization, Algebraic curves, Resolvents, Ritt–Wu's decomposition algorithm, Gröbner bases

1. Introduction

Rational algebraic curves are widely used in geometric modeling and computer graphics and it is recognized that both implicit and parametric representations for rational curves have their inherent advantages: the parametric representation is best suited for generating points along a curve, whereas the implicit representation is most convenient for determining whether a given point lies on a specific curve (Sederberg and Anderson 1984). This motivates the search for a means of converting from one representation to the other. In this paper, we give a complete method of parameterization for algebraic curves in an affine space of any dimension.

In (Abhyankar and Bajaj 1988), a method for computing the genus of plane curves was given, and if genus = 0, they also gave a method for computing the rational parametric equations of the curve. A natural way for parameterizing a space curve is first to find a plane curve which is birational to the space curve and then a set of parametric equations for the space curve can be found if we can find a set of

* The work reported here was supported in part by the NSF Grants CCR-8702108 and 9117870
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parametric equations for the plane curve. In (Abhyankar and Bajaj 1989), this has been done for a special class of space curves, i.e., space curves which can be represented by transversal intersection of two surfaces.

On the other hand, it is a well known result in algebraic geometry that an irreducible variety is birational to a hypersurface (Hartshorne 1977). In particular, an irreducible algebraic curve is birational to an irreducible plane curve. However, we need a constructive method for calculating that irreducible plane curve to solve the general parameterization problem for arbitrary algebraic curves. Such a constructive method implicitly exists in a classic book of Ritt (Ritt 1954). In this paper, based on Ritt's concept of resolvents, we give algorithms of constructing a hypersurface which is birational to a given irreducible variety. Birational maps between the hypersurface and the variety can also be given. Our algorithms for constructing resolvents are different from Ritt's algorithm in two aspects. First, the input of our algorithms is a set of generators of an ideal, while the input of Ritt's algorithm is an irreducible characteristic set of a prime ideal. Second, our algorithms use Ritt–Wu's decomposition algorithm (Wu 1984).

In the case of algebraic curves, this implies that for an irreducible algebraic curve \( C \), we can construct a plane curve which is birational to \( C \). Thus, to find a set of parametric equations for \( C \) we only need to find a set of parametric equations for the plane curve. Such an algorithm has been given in (Abhyankar and Bajaj 1988). In this paper, we present a new and more direct algorithm which does not need to compute the genus of the plane curve. Our method is based on the existence of proper parametric equations for a rational plane curve. The most difficult step in our method is the solution of a set of nonlinear polynomial equations.

The implementation of the algorithms in this paper is based on our implementation of Ritt–Wu's decomposition algorithm a detailed description of which can be found in (Wu 1984) or in (Chou and Gao 1990) for an improved version.

Previous work on finding a plane curve which is birational to a space curve usually use the technique of taking a projection of the space curve into a randomly selected direction and verifying that it is one-to-one (Abhyankar and Bajaj 1989; Garrity and Warren 1989; Kalkbrener 1990). The resolvent method used in this paper is similar to the projection method and deals with the most general case. Also, reducible varieties can be treated by our method, e.g. see Example 5.2.

This paper is organized as follows. In Sect. 2, we introduce some basic notations and notions necessary for the rest of this paper. In Sect. 3, we present methods of constructing a resolvent for a prime ideal. In Sect. 4, we present our method of parameterization for a plane curve. In Sect. 5, we consider the applications to space curves.

2. Preliminaries

Let \( K \) be a computable field of characteristic zero and \( K[x_1, \ldots, x_n] \) or \( K[x] \) be the ring of polynomials in the indeterminates \( x_1, \ldots, x_n \). Unless explicitly mentioned otherwise, all polynomials in this paper are in \( K[x] \).

Let \( P \) be a polynomial. The class of \( P \), denoted by class\((P)\), is the largest \( p \) such that some \( x_p \) actually occurs in \( P \). If \( P \in K \), class\((P) = 0 \). Let a polynomial \( P \) be of class \( p > 0 \). The coefficient of the highest power of \( x_p \) in \( P \) considered as a polynomial