The algebra of majority consensus

HANS-JÜRGEN BANDELT AND Gerasimos C. MELETIOU

Abstract. In this note we show that a median algebra can be defined in terms of a single \( n \)-ary operation for any \( n \geq 5 \), so that besides symmetry and a majority condition only one additional identity is required. This provides a short axiomatic characterization of majority consensus for taxonomic structures.

For a number of taxonomic models the majority rule constitutes a feasible method in order to obtain a consensus for partially conflicting models on a given set of taxonomic units. This includes taxonomic models such as “\( n \)-trees” (labelled rooted trees), “phylogenetic trees” (labelled unrooted trees), weak orders, and the like. In all these situations a taxonomic model can be regarded as a system of certain subsets, partitions, or pairs of taxonomic units, respectively. If \( X_1, \ldots, X_n \) is a profile of such models, then the “majority term”

\[
\bigvee_{I \subseteq \{1, \ldots, n\}, \frac{n}{2} < |I| \leq \frac{n}{2} + 1} \bigwedge_{i \in I} X_i
\]

is the model proposed by majority consensus – provided that the set \( M \) of all models under consideration can be organized as a partial lattice in which all “majority terms” exist. Existence is guaranteed, for instance, when \( M \) is a median semilattice – as is the case for the above mentioned types of taxonomic models; cf. Barthélémy, Leclerc and Monjardet (1986). The majority rule is closely related to the median procedure in median semilattices, e.g., majority consensus and median consensus coincide for every profile of odd size. Barthélémy and Janowitz (1991) have axiomatized the median procedure in median semilattices (see Barthélémy and McMorris (1986) for the particular case of \( n \)-trees).
In this paper we wish to axiomatize the majority consensus for profiles of any fixed size \( n \geq 5 \) by purely algebraic identities, so that we are in the realm of universal algebra. In particular, we are interested in the question of when the system of taxonomic models under consideration receives a median semilattice structure. One should keep in mind that every median semilattice gives rise to a ternary algebra, called a median algebra, where the fundamental operation applied to elements \( x, y, z \) can be interpreted as the median consensus or majority consensus for the profile \( x, y, z \). A median algebra can be defined (without reference to median semilattices) as a ternary algebra satisfying the following three identities (see Bandelt and Hedliková (1983) for a survey on median algebras):

\[
(uvw) = (vwu) = (wuv),
\]

\[
(uvw) = u,
\]

\[
(vwxyz) = (xvw)(wyz),
\]

where the ternary operation is simply written as \( x, y, z \mapsto (xyz) \). This bracket notation will also be used for the \( n \)-ary operations \( f \) studied in the sequel. So, \( f(x_1, \ldots, x_n) \) is expressed as \((x_1 \cdots x_n)\), and if some of the entries are identical, say \( x_1 = \cdots = x_k = x \ (k \geq 0)\), then we write \((x^k x_{k+1} \cdots x_n)\) instead of \((x_1 \cdots x_n)\).

Here is our main result.

**THEOREM 1.** Let \( x_1, \ldots, x_n \mapsto (x_1 \cdots x_n) \) be a symmetric \( n \)-ary operation \((n \geq 5)\) on a set \( M \) such that the following two identities hold for some integer \( s \) with \( \frac{1}{2} n < s \leq \frac{3}{2} n \):

\[
(w_1 \cdots w_{n-s} x^s) = x,
\]

\[
(w_1 \cdots w_{n-1} (x_1 \cdots x_n)) = ((w_1 \cdots w_{n-1} x_1) \cdots (w_1 \cdots w_{n-1} x_n) x_{n+1} \cdots x_n)
\]

for all \( w_i, x_i, x \in M \). Then \( M \) is a median algebra with respect to the ternary operation \( x, y, z \mapsto (xyz) \) defined by

\[
(xyz) := (x^a y^b z^r)^{2s-n}.
\]

**Proof.** To establish symmetry of the ternary operation \( x, y, z \mapsto (xyz) \) requires a little effort. As a prerequisite we need an auxiliary identity:

\[
(w^h x^m y^{n-m-h}) = (w^{h-1} x^{m+1} y^{n-m-h}) \quad \text{for } w, x, y \in M,
\]