A NEW APPROACH TO THE CONVOLUTION OPERATOR ON A FINITE INTERVAL

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The convolution operator on a finite interval defined on a space of $L^2$ functions is studied by relating it to a singular integral operator acting on a space of functions defined on a system of two parallel straight lines in the complex plane $\mathbb{C}$. The approach followed in the paper applies both to the case where the Fourier transform of the kernel functions is an $L^\infty$ function and to the case where the kernel function is periodic, thus yielding a unified treatment of these two classes of kernel functions. In the non-periodic case it is possible, for a special class of kernel functions, to study the invertibility property of the operator giving an explicit formula for the inverse. An example is presented and generalizations are suggested.

1 INTRODUCTION

In the present paper we study the convolution operator on a finite interval $I = [-1, 1]$ which may be written in the form

$$(M \varphi)(x) = \int_{-1}^{1} g(x-t) \varphi(t) dt, \quad x \in ]-1, 1[$$

where $\varphi$ is considered in $L^2(I)$ and $g$ is a temperate distribution assumed to have a Fourier transform $\hat{g}$ in $L^\infty(\mathbb{R})$. The properties of the above operator acting in Sobolev spaces and spaces of Bessel potentials were studied in [1], [3], [4] and [6] by relating it to a singular integral operator on the space $L^2_\mathbb{C}(\mathbb{R}^+)$ of $\mathbb{C}^2$-valued functions that are Fourier transforms of functions with support in $\mathbb{R}^+$. The objective of this paper is to relate the operator $M$ with a singular integral operator acting on a space of functions defined on a contour $\Gamma$ consisting of two parallel straight lines in the complex plane. This brings new features into the formulation and yields a theory that applies also to the convolution operator with periodic kernel function dealt with in [2]. The operator $M$ is shown to be equivalent to a Toeplitz operator acting on a closed subspace of the space of $L^2$ functions defined on
\( \Gamma \). With an additional condition on \( \hat{g} \), \( M \) is equivalent to a standard Toeplitz operator on the Hardy space \((L^2_\mathcal{C}(\Gamma))^2\) of vector valued functions having an analytic extension into the interior of \( \mathcal{C} \setminus S_a \) where \( S_a \) is the strip bounded by \( \Gamma \).

Besides giving a unifying formulation of the above-mentioned two classes of finite interval convolution operators, it is hoped that the new formulation will make it easier to study the invertibility property for these operators. A first step towards this objective is taken in Section 7 by studying the invertibility of a class of operators which appear naturally in the analysis.

The paper is organized as follows. In Section 2 we give some preliminary definitions and results concerning the space \( L_2(\Gamma) \). Section 3 is devoted to the study of a special subspace of \( L_2(\Gamma) \) which plays an important role in the analysis; the properties of some projection operators related to this subspace are also studied in this section. Section 4 contains the derivation of an integral representation for a Toeplitz type operator which is shown in Section 5 to be equivalent to the finite interval convolution operator \( M \). In Section 6 we show that the theory of Sections 4 and 5 also applies to the convolution operator with a periodic kernel function. Finally Section 7 is devoted to the invertibility study for convolution operators whose kernel functions belongs to a certain class which, in a sense, appears naturally in the analysis. Section 8 includes a generalization to Sobolev spaces of the results of the preceding sections and in Section 9 an example is completely studied.

2 PRELIMINARIES

In all that follows, \( \Gamma \) denotes the contour \(-\Gamma_1 + \Gamma_2\) where \( \Gamma_1 \) and \( \Gamma_2 \) are the paths defined by

\[
\Gamma_1 : \mathbb{R} \to \mathbb{C}, \quad t \mapsto ai + t
\]

\[
\Gamma_2 : \mathbb{R} \to \mathbb{C}, \quad t \mapsto -ai + t \quad (a > 0).
\]

We shall use the same symbol to denote the path and the curve defined by its image.

If \( S_\Gamma \) is the singular operator on \( \Gamma \), then two complementary projections \( P^- \) and \( P^+ \) can be defined by

\[
P^- = \frac{1}{2} (I_\Gamma - S_\Gamma)
\]

\[
P^+ = \frac{1}{2} (I_\Gamma + S_\Gamma).
\]

These projections yield the following decomposition for the space \( L_2(\Gamma) \):

\[
L_2(\Gamma) = L^+_2(\Gamma) \oplus L^-_2(\Gamma).
\]