COMPOSITION OPERATORS ON ORLICZ SPACES

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In this paper we characterize the composition operators on Orlicz spaces and study some of their properties.

1 Introduction and preliminaries

Let \( \phi : [0, \infty) \rightarrow [0, \infty) \) be a continuous convex function which satisfies the following conditions

(i) \( \phi(x) = 0 \) iff \( x = 0 \)

(ii) \( \lim_{x \rightarrow \infty} \phi(x) = \infty \)

Such a function is called a Young function. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space. Let

\[
L^\phi(\mu) = \{ f : X \rightarrow \mathbb{C} \text{ measurable : } \int_X \phi(\alpha |f|) \, d\mu < \infty \text{ for some } \alpha > 0 \}
\]

The space \(L^\phi(\mu)\) is called an Orlicz space and is a Banach space with respect to norm

\[
\|f\|_\phi = \inf\{\epsilon > 0 : \int_X \phi\left(\frac{|f|}{\epsilon}\right) \, d\mu \leq 1\}.
\]

If \(\phi(x) = x^p\), \(1 \leq p < \infty\), then \(L^\phi(\mu) = L^p(\mu)\), the space of Lebesgue integrable functions [5]. All functional equations and set relations are taken modulo sets of \(\mu\) - measure 0.

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Simple functions are not necessarily dense in $L^\phi(\mu)$. But, if $\phi$ satisfies $\Delta_2$ - condition (i.e., $\phi(2x) \leq C\phi(x)$, $x > 0$, $C > 0$ constant), then simple functions are dense in $L^\phi(\mu)$. For details about Orlicz spaces we refer to ([2], [4]).

Let $Y \subseteq X$ be a measurable subset of $X$ and $T$ be a measurable transformation of $Y$ into $X$. We define the linear transformation $C_T$ from $L^\phi(\mu)$ into the space of all complex valued measurable functions on $X$ as

$$(C_Tf)(x) = \begin{cases} f(T(x)) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

for all $f \in L^\phi(\mu)$.

If $C_T$ is a bounded linear operator from $L^\phi(\mu)$ into $L^\phi(\mu)$, we say that $C_T$ is a composition operator on $L^\phi(\mu)$ induced by $T$.

For systematic study of composition operators on different functions spaces we refer to [1], [3] and [7]. We end this section with the following definition.

**Definition (A)** Let $N$ and $M$ be two normed linear spaces. Then a map $A : N \rightarrow M$ is said to be non-expansive if $\| T(x) - T(y) \| \leq \| x - y \|$ for all $x, y \in N$.

## 2 Characterizations

The following theorem shows the conditions under which $T$ induces a bounded composition operator on $L^\phi(\mu)$.

**Theorem 2.1** Let $T : X \rightarrow X$ be a measurable mapping. Then it induces a composition operator $C_T$ on $L^\phi(\mu)$, on a $\sigma$-finite space $(X, A, \mu)$, iff there is a constant $M > 1$ such that $\mu(T^{-1}(E)) \leq M\mu(E)$, for all $E \in A$.

**Proof** Let $C_T$ be a composition operator induced by $T$, and $E \in A$ be such that $\chi_E \in L^\phi(\mu)$ (so $\mu(E) < \infty$). Then $\| C_T\chi_E \|_\phi \leq M \| \chi_E \|_\phi$, $M > 0$ and can be taken $\geq 1$. Using the evaluation of the norms, one gets

$$\left[ \phi^{-1}\left( \frac{1}{\mu(T^{-1}(E))} \right) \right]^{-1} = \| C_T\chi_E \|_\phi \leq M \| \chi_E \|_\phi = M \left[ \phi^{-1}\left( \frac{1}{\mu(E)} \right) \right]^{-1}. \quad (1)$$

Since $\phi$ is a continuous convex function satisfying $\phi(x) = 0$ iff $x = 0$ and $\phi(x) < \infty$ for $0 \leq x < \infty$, one has for $x_1 < x_2$,

$$\phi(x_2) - \phi(x_1) = \int_{x_1}^{x_2} \phi'(t)dt \geq (x_2 - x_1)\phi'(x_1) > 0,$$