CELLULAR-INDECOMPOSABLE SUBNORMAL OPERATORS III

Robert F. Olin and James E. Thomson

A bounded operator \( T \) is called cellular-indecomposable if \( L \cap M \neq \{0\} \) whenever \( L \) and \( M \) are nonzero invariant subspaces for \( T \). We prove that a cyclic subnormal operator is cellular-indecomposable if and only if it is quasi-similar to an analytic Toeplitz operator whose symbol is a weak-star generator of \( H^\infty \). This completes our previous work \([6]\), \([6]\).

1. INTRODUCTION.

For a positive measure \( \mu \) with compact support in the complex plane let \( P^2(\mu) \) and \( P^\infty(\mu) \) denote the closures of the analytic polynomials in \( L^2(\mu) \) in the norm topology and in \( L^\infty(\mu) \) in the weak-star topology, respectively. Let \( S_\mu \) denote the operator of multiplication by the independent variable on \( P^2(\mu) \). Each cyclic subnormal operator is unitarily equivalent to some \( S_\mu \) \([1]\). Let \( D \) denote the open unit disk and let \( m \) denote normalized Lebesgue measure on \( \partial D \). We say that \( P^\infty(\mu) = H^\infty(D) \) if \( \text{supp}(\mu) \subseteq \overline{D}, \mu|\partial D << m \), and the natural map of the Hardy space \( H^\infty \) into \( P^\infty(\mu) \) is a surjective linear isometry and weak-star homeomorphism.

Each function \( f \) in \( H^\infty \) induces a natural multiplication operator \( T_f \) on the Hardy space \( H^2 \). The operator \( T_f \) is called the \textit{analytic Toeplitz operator} with symbol \( f \). The function \( f \) is a \textit{weak-star generator} of \( H^\infty \) if the algebra generated by \( f \) and \( 1 \) is weak-star dense in \( H^\infty \).

Our main theorem is the following.

**Theorem 1.** A cyclic subnormal operator is cellular-indecomposable if and only if it is quasi-similar to an analytic Toeplitz operator whose symbol is a weak-star generator of \( H^\infty \).

In \([6]\) we prove the sufficiency of the quasi-similarity condition. Combining the result from page 608 of \([6]\) and Theorem 8 of \([5]\), we see that the necessity is reduced to establishing the following theorem.

**Theorem 2.** Suppose that \( P^\infty(\mu) = H^\infty(D) \) and that the set of analytic bounded point evaluations for \( P^2(\mu) \) equals \( D \). Assume further that \( \mu|\partial D \neq 0 \) and that \( m \) is not absolutely
continuous with respect to $\mu|\partial D$. Then $S_\mu$ is cellular-decomposable.

The most important elements in our previous work were the consequences of our study of factorization and analytic subspaces [4]. The key additional element in this paper is a factorization method developed in [8].

2. PROOFS OF THE THEOREMS.

Before stating our lemmas, we recall some notation from [5], [6], and [8]. If $P_0^\infty(\mu) = H^\infty(D)$, let $P_0^\infty(\mu)$ denote the subspace of functions in $P_0^\infty(\mu)$ that vanish at zero. Let $P_0^\infty(\mu)_*$ denote the predual of $P_0^\infty(\mu)$ with norm $\|\cdot\|_*$. For $a, b \in L^2(\mu)$ define $[a \otimes_2 b]$ in $P_0^\infty(\mu)_*$ by $[a \otimes_2 b](f) = \langle fa, b \rangle$ for each $f$ in $P_0^\infty(\mu)$. For $0 < r < 1$ let $A_r = \{ z : r \leq |z| \leq 1 \}$. For $x \in P^2(\mu)$ let $H_x$ denote the cyclic invariant subspace for $S_\mu$ generated by $x$, i.e., the closed linear span of $\{ z^n x : n = 0, 1, 2, \ldots \}$. The following lemma is an immediate consequence of Lemmas 5 and 6 in [8].

**Lemma 3.** Let $\mu$ be a measure with $P_0^\infty(\mu) = H^\infty(D)$ and let $L \in P_0^\infty(\mu)_*$. Let $\delta, \epsilon, \eta$, and $r \in (0, 1)$ and let $p$ be a positive integer. Let $a, b, c_1, \ldots, c_p \in L^2(\mu)$ and let $d_1, \ldots, d_p \in L^2(\mu|A_r)$. Suppose (i) and (ii) below are true.

(i) The measure $\mu$ is not absolutely continuous with respect to $\mu|\partial D$.
(ii) $\| L - [a \otimes_2 b]\|_* \leq \delta^2$.

Then there exist $x \in P^2(\mu)$ and $y \in L^2(\mu|A_r)$ such that (iii)-(viii) below hold.

(iii) $\| x \| \leq 6\delta$.
(iv) $\| y \| \leq \delta$.
(v) $\| [x \otimes c_j] \|_* < \eta$ for $j = 1, \ldots, p$.
(vi) $\| y d_j \|_1 \leq \eta$ for $j = 1, \ldots, p$.
(vii) $\int_{|x| < \delta} |x|^2 d\mu < \eta$.
(viii) $\| L - (a + x) \otimes (b + y) \|_* < \epsilon$.

**Lemma 4.** Let $\mu$ be a measure as in the statement of Theorem 2. Also assume $S_\mu$ is cellular-indecomposable. If $x \in P^2(\mu)$ and $x = 0$ a.e. on $D$, then $x = 0$ a.e. on $D$.

**Proof of Theorem 4:** Let $x \in P^2(\mu)$ with $x = 0$ a.e. on $D$. Let $\nu$ be the measure with $d\nu = |x|^2 d\mu$. The operators $S_\mu|H_x$ and $S_\nu$ are unitarily equivalent, where the natural unitary arises from the mapping $p \to px$ for polynomials $p$ in $P^2(\nu)$.

The measure $\nu$ is supported on $\partial D$ and the measure $\mu$ is not absolutely continuous with respect to $\nu$ by hypothesis. It thus follows from the F. and M. Riesz theorem that no nonzero function in $L^1(\nu)$ annihilates the polynomials. Hence, $P_0^\infty(\nu) = L^\infty(\nu)$. Thus, $S_\nu$