THE COURANT-FISCHER THEOREM AND THE SPECTRUM OF SELFADJOINT BLOCK BAND TOEPLITZ OPERATORS

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We show that if \( T(F) \) is a selfadjoint block Toeplitz operator generated by a trigonometric matrix polynomial \( F \), then the spectrum of \( T(F) \) as well as the limiting set \( \Lambda(F) \) of the eigenvalues of the truncations \( T_n(F) \) is the union of a finite collection of segments (the spectral range of \( F \)) and at most a finite set of points for which we give an upper bound.

Let \( T \) be the complex unit circle and let \( F \) be a continuous \( N \times N \)-matrix valued function on \( T \), \( F \in C_{N \times N}(T) \). Let \( F_n \in C^{N \times N} \) stand for the \( n \)th Fourier coefficient of \( F \),

\[
F_n = \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) e^{-in\theta} \, d\theta \quad (n \in \mathbb{Z}),
\]

and consider the infinite block Toeplitz matrix \( T(F) = (F_{j-k})_{j,k=0}^\infty \) as well as its truncations \( T_n(F) = (F_{j-k})_{j,k=0}^{n-1} \). We think of \( T(F) \) as an operator on the \( C^N \)-valued \( l^2 \) space over the nonnegative integers. The spectrum \( \sigma(A) \) of an operator \( A \) is defined in the usual way, and by the essential spectrum \( \sigma_{\text{ess}}(A) \) of \( A \) we understand the set of all \( \lambda \in \mathbb{C} \) for which \( A - \lambda I \) is not Fredholm (i.e. not invertible modulo compact operators).

Now suppose \( F \in C_{N \times N}(T) \) assumes only selfadjoint values. Clearly, this is the case if and only if \( F_n = F_{-n} \) for all Fourier coefficients of \( F \). For \( z \in T \), denote by

\[
\lambda_i^F(z) \leq \ldots \leq \lambda_N^F(z)
\]

the eigenvalues of \( F(z) \) counting multiplicities. From the results of \cite[Ch. 2]{4} we infer that for each \( i \in \{1, \ldots, N\} \) the function \( \lambda_i^F : T \to \mathbb{R} \) is continuous. We call \( \lambda_i^F \) the \( i \)th eigenvalue function of \( F \). Put \([a_1, b_1] := \lambda_1^F(T)\). The set

\[
sr(F) := [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_N, b_N]
\]

is referred to as the spectral range of \( F \).

The associated function \( \tilde{F} \) of \( F \) is defined by \( \tilde{F}(z) = F(z^{-1}) \) (\( z \in T \)). It is well known that

\[
\sigma_{\text{ess}}(T(F)) = \sigma_{\text{ess}}(T(\tilde{F})) = sr(F) = sr(\tilde{F})
\]
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(see, e.g., [1, Theorem 2.94] or [3, Theorem 4.3 on p. 575]). The (uniform) limiting set of the eigenvalues of $T_n(F)$ is the set

$$\Lambda(F) := \{ s \in \mathbb{R} : \exists \{ s_n \} \in \sigma(T_n(F)) \text{ such that } s_n \to s \}.$$ 

The following fact, which goes back to Widom [7], [8], is also well known.

**Theorem (Widom).** If $F \in C_{N \times N}(T)$ takes on only selfadjoint values then

$$\Lambda(F) = \sigma(T(F)) \cup \sigma(T(\tilde{F})).$$

**Proof.** It suffices to show that $0 \not\in \Lambda(F)$ if and only if $0 \not\in \sigma(T(F)) \cup \sigma(T(\tilde{F}))$.

If $0 \not\in \Lambda(F)$, then there is an open neighborhood $U \subset \mathbb{R}$ of $0$ such that $U \cap \sigma(T_n(F)) = \emptyset$ for infinitely many $n_k$'s. Since $T_n(F)$ is selfadjoint, this implies that the norms $\|T_n^{-1}(F)\|$ are uniformly bounded, which easily yields the invertibility of $T(F)$. As $T_n(\tilde{F})$ equals $W_n T_n(F) W_n$ where $W_n$ is the $n \times n$ block matrix

$$W_n = \begin{pmatrix}
0 & \cdots & 0 & I \\
0 & \cdots & I & 0 \\
\vdots & \cdots & \cdots & \cdots \\
I & \cdots & 0 & 0
\end{pmatrix},$$

it follows analogously that $T(\tilde{F})$ is also invertible.

Conversely, if $0 \not\in \sigma(T(F)) \cup \sigma(T(\tilde{F}))$, then $\|T_n^{-1}(F)\|$ is uniformly bounded (see, e.g., [2, Theorem VII.5.3.] and [1, Theorem 7.20]), so the eigenvalues of the selfadjoint matrices $T_n(F)$ are bounded away from $0$, whence $0 \not\in \Lambda(F)$. $\blacksquare$

To have a concrete example, note that if $F(z) = \begin{pmatrix} 1 & z \\ z^{-1} & 0 \end{pmatrix}$ then

$$\text{sr}(F) = \left\{ \frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right\}, \quad \sigma(T(F)) = \text{sr}(F) \cup \{1\}, \quad \sigma(T(\tilde{F})) = \text{sr}(F) \cup \{0\}.$$ 

Hence,

$$\Lambda(F) = \left\{ \frac{1 - \sqrt{5}}{2}, 0, 1, \frac{1 + \sqrt{5}}{2} \right\}.$$ 

The purpose of this note is to provide additional information about $\sigma(T(F))$, $\sigma(T(\tilde{F}))$ and thus about $\Lambda(F)$ in the case where $F$ is a trigonometric polynomial with selfadjoint values. Our approach is based on Widom’s theorem and on the following result.

**Theorem (Courant–Fischer).** Let $H_1$ and $H_2$ be selfadjoint $m \times m$ matrices and suppose $H_2 \leq H_1$. Denote by $\mu_1^k \leq \ldots \leq \mu_m^k$ the eigenvalues of $H_k$ ($k = 1, 2$) and let $r$ be the rank of $H_1 - H_2$. Then $\mu_1^2 \leq \mu_i^1$ for $i = 1, \ldots, m$ and $\mu_j^1 \leq \mu_j^{1+r}$ for $j = 1, \ldots, m - r$.

A proof is in [5], for example. $\blacksquare$