REPRODUCING KERNELS FOR HARDY SPACES
ON MULTIPLY CONNECTED DOMAINS

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Associated with a bounded $g$-holed ($g \geq 0$) planar domain $D$ are two types of reproducing kernel Hilbert spaces of meromorphic functions on $D$. We give explicit formulas for the reproducing kernel functions of these spaces. The formulas are in terms of the theta functions defined on the Jacobian variety of the Schottky double of the region $D$. As applications we settle a conjecture of Abrahamse concerning Nevanlinna-Pick interpolation on an annulus and obtain explicit formulas for the curvature (in the sense of Cowen and Douglas) of rank 1 bundle shift operators.

INTRODUCTION.

There are two equivalent $g$-real dimensional tori of Hardy spaces on a $g$-holed planar domain $D$ ($g \geq 1$) with $\partial D$ analytic. One of these tori is the torus of least harmonic majorant Hardy spaces $H^2_\lambda(D)$ parametrized by $\lambda = (\lambda_1, \ldots, \lambda_g)$ in $T^g$ [2,4,13,14]. Roughly the elements in $H^2_\lambda(D)$ are multivalued analytic functions $f$ on $D$ with $|f|^2$ (single-valued) possessing a least harmonic majorant and $f$ experiencing a multiplicative jump by the factor $\lambda_j$ when $f$ is continued around a loop in $D$ homotopic to the boundary of the $j$th hole ($j = 1, \ldots, g$). The second torus of Hardy spaces is the torus $V_a$ of Hardy spaces associated with divisors of measures $m$ on $\partial D \cup \partial D [6,7]$. Every such representing measure is the restriction to $\partial D$ of a non-negative symmetric meromorphic representing differential $\omega$ on the double $X$ of $D$. The space $M_a$ of such representing differentials is a $g$-real-dimensional convex set. The zero divisor of $\omega$ has the form $(\omega)_0 = D(\omega) + JD(\omega)$, where $D = P_1 + \cdots + P_g$ with $P_1, \ldots, P_g$ points in $Y$ and $J : X \to X$ is the usual anti-holomorphic involution on $X$. The non-negative divisors $D$ of degree $g$ satisfying $D + JD = (\omega)_0$ for some $\omega$ in $M_a$ form a $g$-real-dimensional torus $V_a$ in the symmetric product $X^{(g)}$. Given $D$ in $V_a$ with $D + JD = (\omega)_0$, one introduces the Hardy space $H^2_D(\omega)$ which is the closure in $L^2(\omega)$ of the meromorphic functions on $Y$ whose divisors $(f)$ satisfy $(f) + D \geq 0$. The explicit one-to-one correspondence between the two tori $\{H^2_\lambda(D) : \lambda \in T^g\}$ and $\{H^2_D(\omega) : D \in V_a\}$ is given in [7].
All of the above mentioned Hardy spaces are reproducing kernel Hilbert spaces. The main result here is an explicit description of the reproducing kernels $k(\cdot, z, \lambda), z \in D, \lambda \in \mathbb{T}^g$ for $H^2(D)$ and $k(\cdot, z, \mathcal{D}), z \in D \setminus \text{supp } \mathcal{D}, \mathcal{D} \in \mathcal{V}_a, \text{ for } H^2_a(\omega)$. These descriptions are given in terms of theta functions on the marked double $X$.

The basic method is to imbed the marked surface $X$ onto a conformal model $X_0$ in the Jacobian variety $\text{Jac}(X) = \mathbb{C}^g/L_\tau$, where $L_\tau = \mathbb{Z}^g + \tau \mathbb{Z}^g$ is the period lattice of holomorphic one-forms on $X$. Then, for example, for $\mathcal{D}$ in $\mathcal{V}_a$ the reproducing kernel $k(\cdot, z : \mathcal{D})$ of $H^2_a(\omega)$ is given as the meromorphic function

$$k(\zeta, z : \mathcal{D}) = \frac{\theta(\zeta + \bar{z} - a - \bar{a} - t)\theta(t) \theta_t(\zeta + \bar{a})\theta_t(\bar{a} + a)}{\theta(\zeta - a - t)\theta(\bar{z} - \bar{a} - t) \theta_t(\zeta + \bar{a})\theta_t(\bar{a} + a)} , \quad \zeta \in X_0, \quad (0.1)$$

where $\theta$ is the Riemann theta function associated with $L_\tau$, $\theta_t$ a theta function with characteristics of a non-singular odd half-period and $t$ a point in $\mathbb{R}^g/\mathbb{Z}^g \hookrightarrow \text{Jac}(X)$ determined by $\mathcal{D}$. When $\lambda = (1, 1, \cdots, 1)$, the representation of the reproducing kernel $k(\cdot, z : \lambda)$ in the form $(0.1)$ was given by Fay [10].

We mention that McCullough and Shen [11] have recently used theta-function representations for a class of reproducing kernels over the annulus and obtained a set of functional identities satisfied by these kernels. Also, Alpay and Vinnikov [3] have worked with theta-function representations for reproducing kernels for Hilbert spaces of half-order differentials over a real compact Riemann surface while developing analogues of de Branges-Rovnyak model spaces in this context.

The results of this paper on representations of reproducing kernels extend equally well to the setting where the planar domain $D$ is replaced by a finite bordered Riemann surface (i.e. surfaces with handles), as all the machinery concerning the conformal model $X_0$ in the Jacobian variety $\text{Jac}(X)$ and associated theta functions extends to this more general setting. Indeed the result of Fay [10] is given in this level of generality. However there is no canonical choice of rank 1 bundle shift (in the sense of Abrahamse and Douglas [2]) in this more general context, since there is no global uniformizing variable $z$ as in the planar case.

Much can be learned from the concrete form $(0.1)$. Here we indicate several applications. The continuous dependence of $k(\zeta, z : \lambda)$ as a function of $\lambda$ in $\mathbb{T}^g$ is fairly transparent from the theta function description of $k(\zeta, z : \lambda)$. This dependence was noted first by Widom [14]. The reproducing kernels $k(\zeta, z : \lambda), \lambda \in \mathbb{T}^g$, were used by Abrahamse [1] to provide a solution to the Nevanlinna-Pick problem on multiply connected domains. The explicit description of $k(\zeta, z : \lambda)$ obtained here can be used to answer, when $g = 1$, a question of Abrahamse [1] on the Nevanlinna-Pick problem. The representation $(0.1)$ makes it easy to analyse for $\mathcal{D}$ fixed the zero divisor $\mathcal{D}_z$ of $k(\cdot, z : \mathcal{D})$. We show how $z \rightarrow D_z$ can be nicely