FREDHOLM COMPOSITION OPERATORS ON
SPACES OF HOLOMORPHIC FUNCTIONS

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Composition operators on vector spaces of holomorphic functions are considered. Necessary conditions that range of the operator is of a finite codimension are given. As a corollary of the result it is shown that a composition operator $C_\varphi$ on a certain Banach space of holomorphic functions on a strictly pseudoconvex domain with $C^2$ boundary or a polydisc or a compact bordered Riemann surface or a bounded domain $D$ such that $\text{int}\bar{D} = D$ is invertible if and only if it is a Fredholm operator if and only if $\varphi$ is a holomorphic automorphism.

In his thesis [4] Schwartz showed that a composition operator $C_\varphi$ on the Hardy space $H^2$ on the unit disc is invertible if and only if $\varphi$ is a conformal automorphism of the unit disc. Cima, Thomson and Wogen [2] proved that only the invertible operators on $H^2$ are Fredholm. Thus we see that a composition operator $C_\varphi$ on the Hardy space $H^2$ is invertible if and only if it is a Fredholm operator if and only if $\varphi$ is a holomorphic automorphism on the unit disc. Bourdon [1] has shown that similar results hold for various Banach spaces of analytic functions on the unit disc. In this paper we consider the problem involving the composition operators $C_\varphi$ which satisfies the condition $\dim A/C_\varphi(A) < \infty$ on a space $A$ of holomorphic functions on a complex manifold. The condition that $\dim A/C_\varphi(A) < \infty$ is a necessary condition that $C_\varphi$ is a Fredholm operator, that is, a Fredholm operator $C_\varphi$ satisfies the condition $\dim A/C_\varphi(A) < \infty$. Thus we consider a class of composition operators which contains Fredholm operators. In general an invertible operator on a Banach space
is a Fredholm operator, but there are invertible composition operators \( C_\varphi \) on a space of holomorphic functions such that \( \varphi \) is not a holomorphic automorphism.

**EXAMPLE 1.** Let \( \psi(z) = \frac{2z+1}{2z+2} \) and \( z_n = \psi \circ \cdots \circ \psi(0) \), the \( n \)-times composition of \( \psi \) of 0 for every positive integer \( n \). Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \setminus \{ z_n \}_{n=1}^\infty \), \( A = H^2|D| \), and \( \varphi = \psi^{-1}|D \). Then \( \varphi(D) = D \setminus \{ 0 \} \) and the space \( A \circ \varphi = \{ f \circ \varphi : f \in A \} \) coincides with \( A \).

**EXAMPLE 2.** Let \( D = \{ z \in \mathbb{C} : \Re z > 0, |z| < 1 \} \), \( \varphi(z) = \frac{2z+1}{2z+2} \) on \( D \), and \( A = \mathcal{H}_2|D| \), where \( C(D) \) denotes the space of complex valued continuous functions on \( D \). Then the space \( A \circ \varphi = \{ f \circ \varphi : f \in A \} \) coincides with \( A \). We also see that there is an open neighborhood \( V_0 \) of the origin such that \( \varphi \) is extended as an analytic function on \( V_0 \cup D \) into \( D \).

These examples show that \( \varphi \) need not be a surjection even if \( C_\varphi \) is invertible. Two examples above are simple and might be seen to be pathological, but the phenomenon in these examples happen for a variety of spaces of analytic functions as we see in Theorem 1. In this paper we consider the problem involving the connection between the domain and the vector space of functions. The main result in this paper is Theorem 1. As corollaries of the theorem we see that for kinds of Banach spaces of analytic functions a composition operator \( C_\varphi \) is invertible if and only if it is a Fredholm operator if and only if \( \varphi \) is a holomorphic automorphism.

**THEOREM 1.** Let \( \Omega \) be a complex manifold of dimension \( n \) and \( D \) a relatively compact domain in \( \Omega \). Suppose that \( A \) is a subspace of the space \( H(D) \) of all holomorphic functions on \( D \) which satisfies the following two conditions:

1. there is a sequence \( \{ F^{(i)} \} \) in \( A^n = \{(f_1, \ldots, f_n) : f_k \in A \} \) such that each \( F^{(i)} \) can be holomorphically extended to \( \Omega \), and that for every linear combination \( \sum a_iF^{(i)} \) of \( \{ F^{(i)} \} \) there is an open set \( U \) in \( D \) such that \( \sum a_iF^{(i)} \) is univalent on \( U \);
2. the equation \( \dim A|K = \sharp K \) holds for every finite subset \( K \) of \( D \), where \( \sharp K \) denotes the number of elements in \( K \).

Suppose that \( \varphi \) is a holomorphic map from \( D \) into \( D \) such that \( A \circ \varphi \subset A \) and \( \dim A/A \circ \varphi < \infty \). Then \( \varphi \) is univalent and one of the following holds: