SCOTT BROWN'S TECHNIQUES FOR PERTURBATIONS OF DECOMPOSABLE OPERATORS

H. Mohebi and M. Radjabalipour

Using Scott Brown's techniques, J. Eschmeier and B. Prunaru showed that if \( T \) is the restriction of a decomposable (or \( S \)-decomposable) operator \( B \) to an invariant subspace such that \( \sigma(T) \) is dominating in \( C \setminus S \) for some closed set \( S \), then \( T \) has an invariant subspace. In the present paper we prove various invariant subspace theorems by weakening the decomposability condition on \( B \) and strengthening the thickness condition on \( \sigma(T) \).

0. INTRODUCTION. Let \( B \) be a bounded linear operator on a Banach space \( Z \) and let \( G \) be a nonempty open subset of \( C \). We say \( B \) is decomposable in \( G \) if for every open subset \( H \) of \( G \) there exists an invariant subspace \( M \) of \( B \) such that

\[
\sigma(B|_M) \subset \overline{H} \quad \text{and} \quad \sigma(B/M) \subset C \setminus H.
\]

(Here \( B/M \) denotes the operator induced by \( B \) on \( Z/M \).) The class of such operators was initially studied by C. Foias and later generalized and investigated by others. (See [2, 3, 7, 9, 10, 11, 12, 17, 20] and the references cited there.) It is shown in [9] that if \( B \) is decomposable in \( G \), if \( T \) is the restriction of \( B \) to an invariant subspace \( X \), and if \( \sigma(T) \) is dominating in \( G \), then \( T \) has a (nontrivial) invariant subspace. (A set \( K \) is dominating in a nonempty open set \( G \) if \( \sup\{|f(z)| : z \in G\} = \sup\{|f(z)| : z \in K \cap G\} \) for all \( f \in H^\infty(G) \).) This result of [9] is an extension of the works initiated by S. Brown and generalized by others. (See [2, 5, 6, 9] and the references cited there.) In the present paper, we study the invariant subspaces of the restrictions \( T \) of operators \( B \) satisfying (*) for a smaller family of open sets \( H \subset G \). However, in some cases we assume \( \sigma(T) \cap G \) is thicker. In fact, we will prove three main invariant subspace theorems whose hypotheses are of the following types. The first type is an extension of, and mainly inspired by, the techniques of [9]. The hypothesis consists of the following conditions (1)-(3).

(1) The operators \( T \in B(X) \) and \( B \in B(Z) \) satisfy \( qT = Bq \) for some injective \( q \in B(X, Z) \)

\(^1\) The research is supported by a grant from the Institute for Studies in Theoretical Physics and Mathematics (IRAN).
with a closed range $qX$.

(2) There exist sequences $\{G(n)\}$ of open sets and $\{M(n)\}$ of invariant subspaces of $B$ such that $G(n) \subset G(n+1)$, $G = \bigcup_n G(n)$, $\sigma(B|M(n)) \subset C\setminus G(n)$ and $\sigma(B|M(n)) \subset G(n)$, $n = 1, 2, \ldots$.

(3) The spaces $X$ and $Z$ are reflexive and $\sigma(T) \setminus \sigma_p(B)$ is dominating in $G$.

Condition (1) implies that $T$ is similar to the restriction of $B$ to its invariant subspace $qX$. Condition (2) is much weaker than the decomposability condition on $B$. In (3), $\sigma_p(B)$ denotes the point spectrum of $B$. Condition (3), in particular, implies that $\sigma(T)$ is dominating in $G$.

The second type is inspired by [9;13] and assumes (1) and the following conditions (2') and (3').

(2') There exist sequences $\{\Delta(n)\}$ of open sets and $\{M_1(n)\}$ and $\{M_2(n)\}$ of invariant subspaces of $B$ such that $\Delta(n) \subset \Delta(n+1)$, $G = \bigcup_n \Delta(n)$, $\sigma(B|M_1(n)) \subset \Delta(n)$, $\sigma(B|M_2(n)) \subset C \setminus \Delta(n+1)$, $M_1(n) + M_2(n)$ is closed, and $\sigma(B/[M_1(n) + M_2(n)]) \subset \Delta(n+1) \setminus \Delta(n)$, $n = 1, 2, \ldots$.

(3') For each $n$ there exists $n' > n$ such that no component of $G \setminus \sigma(T)$ intersects the boundaries of $\Delta(n)$ and $\Delta(n')$ simultaneously.

Again here condition (2') is weaker than the decomposability condition on $B$. Condition (3'), in particular, implies that $\sigma(T)$ is dominating in $G$.

In the third type we are inspired by [2;6] and assume $T$ and $B$ satisfy condition (1) and the following conditions (2'') and (3'').

(2'') Condition (2') is satisfied and

$$\sup\{||y||/||x+y|| : y \in M_1(n), x \in M_2(n); n = 1, 2, \ldots\} = d < \infty$$

(3'') $\sigma(T)$ is dominating in $G = \bigcup_n \Delta_n$.

Condition (2'') is weaker than the unconditional decomposability of $B$ assumed in [2]. (See also [18] for similar conditions.)

Although the main results of [2,6] (and others) can be obtained as special cases of [9], the three extensions of the present paper are basically different and are not special cases of each other. To