The automorphism group of a function lattice: A problem of Jónsson and McKenzie

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Abstract. It is shown that $\text{Aut}(L^Q)$ is naturally isomorphic to

$$\text{Aut}(L) \times \text{Aut}(Q)$$

when $L$ is a directly and exponentially indecomposable lattice, $Q$ a non-empty connected poset, and one of the following holds: $Q$ is arbitrary but $L$ is a jm-lattice, $Q$ is finitely factorable and $L$ is complete with a join-dense subset of completely join-irreducible elements, or $L$ is arbitrary but $Q$ is finite. A problem of Jónsson and McKenzie is thereby solved. Sharp conditions are found guaranteeing the injectivity of the natural map $\nu_{P,Q}$ from $\text{Aut}(P) \times \text{Aut}(Q)$ to $\text{Aut}(P^Q)$ ($P$ and $Q$ posets), correcting misstatements made by previous authors. It is proven that, for a bounded poset $P$ and arbitrary $Q$, the Dedekind-MacNeille completion of $P^Q$, $\text{DM}(P^Q)$, is isomorphic to $\text{DM}(P)^Q$. This isomorphism is used to prove that the natural map $\nu_{P,Q}$ is an isomorphism if $\nu_{\text{DM}(P),Q}$ is, reducing a poset problem to a more tractable lattice problem.

1. Introduction

The function space $P^Q$, where $P$ and $Q$ are ordered sets, is the poset of order-preserving maps from $Q$ to $P$ ordered pointwise. If $L$ is a lattice, $L^Q$ is a function lattice. A poset $R$ is directly indecomposable if $P \times Q \cong R$ implies precisely one of $P$ and $Q$ is trivial. It is exponentially indecomposable if $P^Q \cong R$ implies $Q$ is trivial. The automorphism group of $R$ is denoted $\text{Aut}(R)$. Jónsson and McKenzie ask in [19, Problem 12.3] if

$$\text{Aut}(L^Q) \cong \text{Aut}(L) \times \text{Aut}(Q)$$

Presented by B. Jónsson.
Received June 20, 1994; accepted in final form May 2, 1995.
1991 Mathematics Subject Classification. 06A23, 06B10, 06F15, 08A35, 08B26.

Keywords and phrases. function lattice, automorphism group, exponentiation, logarithm, strict refinement property, jm-lattice, ideal completion, Dedekind-MacNeille completion.

The author would like to thank his supervisor, Dr. H. A. Priestley, for her direction and advice as well as his undergraduate supervisor, Prof. Garrett Birkhoff, and Dr. P. M. Neumann for comments regarding the paper. This material is based upon work supported under a (U.S.) National Science Foundation Graduate Research Fellowship and a Marshall Aid Commemoration Commission Scholarship.
when $L$ is a directly and exponentially indecomposable lattice, and $Q$ a non-empty connected poset, which, possibly, is finite. We are working towards the most general conditions for which the answer is yes.

There is in general a trade-off between conditions on the base and conditions on the exponent. In §5 (Theorem 5.6) we handle the case of arbitrary exponent (subject to the other hypotheses), but we assume the base is a $jm$-lattice (a complete lattice whose completely join-irreducible elements form a join-dense subset, and dually for meet). We vastly generalize Duffus and Wille’s result for finite exponents and bases of finite length ([14, Theorem 1]), as every such base is a $jm$-lattice. They use the “scaffolding,” a representation valid only for lattices of finite length analogous to Priestley’s for distributive lattices ([24]), so their technique is inherently limited. We extend Markowsky’s representation for $jm$-lattices by bipartite directed graphs ([22, Definition 1.3(d)]) to obtain what Duffus and Rival call a “logarithmic property” ([13, §1]).

In §7, we handle the case of arbitrary base, but we assume the exponent is finite (Theorem 7.7). We at any rate solve the problem above posed by Jönsson and McKenzie. We generalize their own theorem which only takes care of subdirectly irreducible lattices ([19, Theorem 11.5]).

Our trick is to take the ideal completion of $L^Q$ repeatedly (see [20, §II]). By [12, Theorem 3.1], the operation of completing by ideals commutes with exponentiating by $Q$, so we eventually get the dual of an algebraic lattice, which therefore has a join-dense subset of completely join-irreducible elements ([4, Theorem 8.8.16]). In other words, we get a hybrid of the function lattices dealt with in §§5 and 7. The base is a $j$-lattice, “half” of a $jm$-lattice: it is complete and the completely join-irreducible elements are join-dense. It turns out that, for such a base, the exponent need not be finite, but merely finitely factorable, a product of finitely many directly indecomposable posets (Theorem 6.18). [Although independently arrived at, our method resembles Novotny’s ([23, §§7 and 8]) and Bauer’s ([1, §5]). The former analyzed function lattices with totally ordered bases. The latter looked at a variant of the above problem but assumed that the ideal lattice of the base was directly and exponentially indecomposable ([1, Satz 6.5.3]).]

The fact we can expand the class of exponents if we restrict the class of bases illustrates the trade-off mentioned earlier, in more ways than one: Jönsson ([18, Theorem 14]) has shown that we may replace the complete lattice by a bounded poset (yielding a $j$-poset), provided we assume the finitely factorable exponent satisfies the ascending chain condition. Indeed, Jönsson proves that the “natural” map

$$v_{P,Q} : \text{Aut}(P) \times \text{Aut}(Q) \rightarrow \text{Aut}(P^Q)$$