We describe all solutions of the two-sided tangential interpolation problem in the class of matrix-valued Hardy functions when symmetries are added: these symmetries are defined in terms of involutions of $H^2$. The obtained results are applied to a one-sided two-points tangential interpolation for matrix functions.

1 Introduction

In the present paper we consider an interpolation problem in a certain subspace of the Hilbert space $H^{p \times q}$ of $p \times q$ matrices with entries in the Hardy space $H^2$ of the open unit disk $D$. The inner product in $H^{p \times q}$ is defined by

$$
\langle H, G \rangle_{H^{p \times q}} = \frac{1}{2\pi} \int_0^{2\pi} \text{Trace} \left( G(e^{it})^* H(e^{it}) \right) dt.
$$

(1.1)

Definition 1.1 We denote by $H^{p \times q}(J_1, J_2, s)$ the class of all functions $H \in H^{p \times q}$ which satisfy the symmetry relation

$$
J_1 H(s(z)) J_2 = H(z),
$$

(1.2)

where $J_1 \in \mathbb{C}^{p \times p}$ and $J_2 \in \mathbb{C}^{q \times q}$ are two given signature matrices: $J_1 = J_1^* = J_1^{-1}$, and where $s(z)$ is a conformal involutive mapping from $D$ onto itself which is given in the form

$$
s(z) = \frac{\omega - z}{1 - \overline{z} \omega}, \quad \omega \in D.
$$

(1.3)

It is easily seen that except of the case $s(z) = z$, formula (1.3) presents the general form of a conformal involutive mapping from $D$ onto itself and that $H^{p \times q}(J_1, J_2, s)$ is a subspace of $H^{p \times q}$. In this subspace we consider the two-sided Nudelman type tangential interpolation problem formulated in the residue form. The data set for this problem is an ordered collection

$$
\Omega = \{C_+, C_-, A_+, A_-, B_+, B_-, \Gamma_1, \Gamma_2\}
$$

(1.4)
of eight matrices $C_+ \in \mathbb{C}^{p \times n_\pi}$, $C_- \in \mathbb{C}^{q \times n_\pi}$, $A_\pi \in \mathbb{C}^{n_\pi \times n_\pi}$, $A_\zeta \in \mathbb{C}^{n_\zeta \times n_\zeta}$, $B_+ \in \mathbb{C}^{n_\zeta \times p}$, $B_- \in \mathbb{C}^{n_\zeta \times q}$ and $\Gamma_1, \Gamma_2 \in \mathbb{C}^{n_\pi \times n_\pi}$ such that

$$\text{spec} A_\zeta \cup \text{spec} A_\pi \subset \mathbb{D}$$

(here and elsewhere in the paper spec$X$ denotes the set of all the eigenvalues of a matrix $X$), and the following Sylvester equalities hold:

$$A_\zeta \Gamma_1 - \Gamma_1 A_\pi = B_- C_- - B_+ C_+$$
$$A_\zeta \Gamma_2 - \Gamma_2 s(A_\pi) = B_- J_2 C_- - B_+ J_1 C_+$$

(1.6)

where $s(A_\pi) = (\omega I_{n_\pi} - A_\pi)(I_{n_\pi} - \bar{\omega} A_\pi)^{-1}$, in accordance with (1.3).

Denote by Res$_D F(z)$ the sum of residues of the meromorphic matrix function $F(z)$ with respect to the points inside the unit disk $\mathbb{D}$; the subscript $\mathbb{D}$ will be often omitted.

**Problem 1.2** Given a set $\Omega$ of matrices (1.4) find all functions $H \in \mathbb{H}_2^{p \times q}(J_1, J_2, s)$ such that

$$\text{Res}_D (z I - A_\zeta)^{-1} B_+ H(z) = B_-$$
$$\text{Res}_D H(z) C_- (z I - A_\pi)^{-1} = C_+$$

(1.7)
$$\text{Res}_D (z I - A_\zeta)^{-1} B_+ H(z) C_- (z I - A_\pi)^{-1} = \Gamma_1$$
$$\text{Res}_D (z I - A_\zeta)^{-1} B_+ H(z) J_2 C_- (z I - s(A_\pi))^{-1} = \Gamma_2$$

(1.8)

(1.9)
(1.10)

Conditions (1.9), (1.10) are added to the left-sided condition (1.7) and the right-sided condition (1.8) to take into account possible intersections of the spectra of the matrices $A_\zeta$, $A_\pi$ and $s(A_\pi)$.

Note that Sylvester equations (1.6) follow immediately from (1.7)-(1.10):

$$A_\zeta \Gamma_1 - \Gamma_1 A_\pi = \text{Res} [A_\zeta (z I - A_\zeta)^{-1} B_+ H(z) C_- (z I - A_\pi)^{-1} - (z I - A_\zeta)^{-1} B_+ H(z) C_- (z I - A_\pi)^{-1} A_\pi]$$
$$= \text{Res} [-B_+ H(z) C_- (z I - A_\pi)^{-1} + z(z I - A_\zeta)^{-1} B_+ H(z) C_- (z I - A_\pi)^{-1}$$
$$+ (z I - A_\zeta)^{-1} B_+ H(z) C_- (z I - A_\pi)^{-1} z - (z I - A_\zeta)^{-1} B_+ H(z) C_- (z I - A_\pi)^{-1} z]$$
$$= -B_+ C_+ + B_- C_-,$$

(1.11)

and the second equality in (1.6) follows analogously from (2.3) in the next section. The equalities (1.6) are therefore necessary conditions for Problem 1.2 to be solvable.

Using the Hilbert structure of $\mathbb{H}_2^{p \times q}$ we will consider the two-sided tangential residue problem under additional norm constraint.

**Problem 1.3** Given a set $\Omega$ of matrices (1.4) and given number $\gamma > 0$ find all functions $H \in \mathbb{H}_2^{p \times q}(J_1, J_2, s)$ satisfying (1.7)-(1.10) and the norm constraint

$$\|H\|_{\mathbb{H}_2^{p \times q}} \leq \gamma.$$