DILATION TO THE UNILATERAL SHIFTS

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The classical result of Foias says that an operator power dilates to a unilateral shift if and only if it is a $C_0$ contraction. In this paper, we consider the corresponding question of dilating to a unilateral shift. We show that for contractions with at least one defect index finite, dilation and power dilation to some unilateral shift amount to the same thing. The only difference is on the minimum multiplicity of the unilateral shift to which the contraction can be (power) dilated. We also obtain a characterization of contractions which are finite-rank perturbations of a unilateral shift, generalizing the rank-one perturbation result of Nakamura.

1. INTRODUCTION

The purpose of this paper is to address the problem, which bounded linear operator on a complex separable Hilbert space can be dilated to a unilateral shift. Recall that an operator $A$ on space $H$ is said to dilate (resp. power dilate) to operator $B$ on $K$ if there is an isometry $V$ from $H$ to $K$ such that $A = V^*BV$ (resp. $A^n = V^*B^nV$ for all $n = 1, 2, \ldots$) or, equivalently, if $B$ is unitarily equivalent to a $2 \times 2$ operator matrix

$$
\begin{pmatrix}
A & * \\
* & *
\end{pmatrix}
$$

with $A$ in its upper left corner (resp. $B^n$ is unitarily equivalent to

$$
\begin{pmatrix}
A^n & * \\
* & *
\end{pmatrix},
$$

under the same unitary operator for all $n = 1, 2, \ldots$). The unilateral shift $S_k$ of multiplicity $k$ ($1 \leq k \leq \infty$) is the operator $S_k(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots)$ on $\sum_{n=0}^{\infty} \oplus H$ with dim $H = k$.

The classical result of Foias settles the corresponding power dilation problem completely: an operator $T$ power dilates to some unilateral shift $S_k$ if and only if $T$ is a contraction ($\|T\| \leq 1$) of class $C_0$, that is, $T$ satisfies $T^n \to 0$ in the strong operator topology, and, moreover, in this case the minimum value of $k$ is $\dim \text{ran} (1 - TT^*)^{1/2}$ (cf. [4, Problem 152]). In this paper, we consider the dilation problem for various classes of operators. In
Section 2 below, we will show that in all the cases we investigated (contractions with at least one defect index finite, $C_0$ contractions, strict contractions, normal contractions and compact contractions) the two classes, one consisting of those which dilate to a unilateral shift and the other those which power dilate, coincide. The only difference is on the minimum multiplicity of the unilateral shift which can be (power) dilated to. In the dilation case, this multiplicity can be as small as 1 if the two defect indices $d_T = \dim \text{ran} \left( (1 - T^*T)^{\frac{1}{2}} \right)$ and $d_{T^*} = \dim \text{ran} \left( (1 - TT^*)^{\frac{1}{2}} \right)$ of the contraction $T$ under consideration are equal, and $d_{T^*} - d_T$ if otherwise. This is in contrast to the multiplicity $d_{T^*}$ of the unilateral shift to which $T$ can be power dilated. Our proof depends on the result of Nakamura [5, Corollary 3] on the rank-one perturbation of unilateral shifts and that of Carey [1, Proposition] on the finite-rank perturbation of isometries.

In Section 3, we take up the problem of characterizing contractions which are finite-rank perturbations of unilateral shifts. We prove in Theorem 3.1 that the completely nonunitary ones among such perturbations are exactly those $C_0$ contractions $T$ with $d_T < d_{T^*}$. This generalizes the rank-one perturbation result of Nakamura [5] although our proof is built upon his.

The monograph [7] by Sz.-Nagy and Foias is our standard reference for the terminology and results of their contraction theory. We will also refer to some basic Fredholm theory from time to time. For this, the reader can consult [2, Chapter XI].

2. UNILATERAL SHIFT DILATION

We say that operator $A$ on $H$ dilates to operator $B$ on $K$ by $n$-dimension $(0 \leq n \leq \infty)$ if $A = V^*BV$ and $\dim (K \ominus VH) = n$ for some isometry $V : H \rightarrow K$.

We start this section with a characterization of operators which dilate to a unilateral shift by finite dimension.

**Theorem 2.1.** An operator $T$ dilates to $S_k$ by $n$-dimension $(1 \leq k \leq \infty, 0 \leq n < \infty)$ if and only if $T$ is a $C_0$ contraction with $d_T < \infty$ and $d_T \neq d_{T^*}$. In this case, $k = d_{T^*} - d_T$ and the minimum value for $n$ is $d_T$.

Note that if $T$ is a $C_0$ contraction, then $d_T \leq d_{T^*}$ always holds (cf. [7, Proposition VI.3.5]).

To prove the necessity part of this theorem, we need the following proposition. Its proof we omit since it is analogous to that of [10, Proposition 3.5]. Recall that a contraction $T$ is of class $C_{11}$ if $T^n x \not\rightarrow 0$ and $T^{*n} x \not\rightarrow 0$ in norm for any nonzero vector $x$. A $C_{11}$ contraction has equal defect indices (cf. [7, Proposition VI.3.5]).