EVERY BANACH ALGEBRA HAS THE SPECTRAL RADIUS PROPERTY

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Nylen and Rodman [NR] introduced the notion of spectral radius property in Banach algebras in order to generalize a classical theorem of Yamamoto on the asymptotic behaviour of the singular values of an \( n \times n \) matrix. In this paper we prove a conjecture of theirs in the affirmative, namely that any unital Banach algebra has the spectral radius property. In fact a slightly more general spectral property holds. We show that for every element which has spectral points which are not of finite multiplicity, the essential spectral radius is the supremum of the set of absolute values of the spectral points that are not of finite multiplicity.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper \( A \) will denote a unital Banach algebra over the complex field \( \mathbb{C} \) and for an element \( a \in A \), \( \sigma(a) \) denotes the spectrum of \( a \) and \( r(a) \) the spectral radius of \( a \). We say \( a \in G(a) \) has finite multiplicity if \( \lambda \) is an isolated spectral point such that the corresponding Riesz idempotent \( e_{\lambda} := \frac{1}{2\pi i} \int_{|\mu - \lambda| = \epsilon} (\mu - a)^{-1} d\mu \) is of finite rank in the sense corresponding Riesz idempotent \( e_{\lambda} := \frac{1}{2\pi i} \int_{|\mu - \lambda| = \epsilon} (\mu - a)^{-1} d\mu \) is of finite rank in the sense of Nylen and Rodman [NR]. The multiplicity of \( \lambda \) equals the rank of \( e_{\lambda} \). Spectral points of finite multiplicity will often be called f.m. spectral points.

Nylen and Rodman [NR] defined an element \( a \in A \) to be of rank one if \( a \neq 0 \) and for every \( b \in A \) there exists \( \lambda \in \mathbb{C} \) such that \( aba = \lambda a \). A rank \( n \) element is an element that can be expressed as a sum of \( n \) rank one elements but not as a sum of less than \( n \) rank one elements. Let \( F \) denote the two-sided ideal of finite rank elements of \( A \) and \( K \) the norm closure of \( F \). For an element \( a \in A \), \( \sigma_K(a) \) and \( r_K(a) \) will respectively denote the essential spectrum of \( a \) and the essential spectral radius of \( a \), that is, the spectrum and spectral radius of \( a + K \) in the quotient Banach algebra \( A/K \). Clearly \( \sigma_K(a) \subseteq \sigma(a) \).

2. THE SPECTRAL RADIUS PROPERTY IN BANACH ALGEBRAS

The spectral radius property is defined [NR] in terms of a spectral point sequence which corresponds to each element of the Banach algebra \( A \).

We outline their construction of the spectral point sequence [NR]. For each \( a \in A \) we consider \( S_1 = \{ \lambda : |\lambda| = r(a) \} \cap \sigma(a) \). If \( S_1 \) consists of f.m. spectral points, we set
$\mu_i(a) = r(a), \ (i = 1, 2, \ldots, n_1)$ where $n_1$ is the sum of the multiplicities of the points of $S_1$. If $S_1$ contains some point which is not of f.m. we set $\mu_i(a) = r(a), \ (i = 1, 2, \ldots)$. In the former case, we proceed by considering $S_2 = \{ \lambda : |\lambda| = r(a_1) \} \cap \sigma(a_1)$, where $a_1 = (1 - e_{S_1})a(1 - e_{S_1})$ and $e_{S_1}$ is the Riesz idempotent corresponding to $S_1$. We view $a_1$ as an element of the Banach algebra $(1 - e_{S_1})A(1 - e_{S_1})$ with unit $1 - e_{S_1}$. If $S_2$ consists of f.m. spectral points, we set 

$\mu_1, j(a) = r(a_1), \ (j = 1, 2, \ldots, n_2)$ where $n_2$ is the sum of the multiplicities of the points of $S_2$. If $S_2$ contains some point which is not of f.m. we set $\mu_1, j(a) = r(a), \ (j = 1, 2, \ldots)$. Continuing this process we obtain a nonincreasing sequence of positive numbers which can either be infinite or finite (we have the latter case if $\sigma(a)$ consists only of f.m. spectral points). We let $n(a)$ denote the length of the sequence. If the sequence is infinite, we denote its limit by $\mu(a)$. We will show that any unital Banach algebra has the following spectral property which was first introduced in [NR]:

For every $a \in A$ for which the spectral point sequence $\{\mu_n(a)\}_{n=1}^{\infty}$ is of infinite length and $\mu(a) = \mu_m(a)$ for some integer $m$, we have that $r_K(a) = \mu(a)$.

The proof is based on a well-established Fredholm theory in a Banach algebra relative to an inessential ideal [BMSW]. We note from [NR] Corollary 2.3, that each element of $F$ has a finite spectrum. By [BMSW] Theorem R.2.6 it follows that $F$ and hence $K$ are inessential ideals of $A$. As an important corollary of the punctured neighbourhood theorem [BMSW] Theorem F.3.9, Barnes, Murphy, Smyth and West obtained very useful results, [BMSW] Theorem R.2.7 and Lemma R.2.3, on the structure of the spectrum of an element in a Banach algebra. Note that Theorem R.2.7 was stated for a semisimple Banach algebra, but it holds without the hypothesis of semisimplicity. We will indicate a short proof of the following special case that will be needed:

**PROPOSITION 2.1** Let $a \in A$ and $\Omega = \{ \lambda \in \sigma(a) : |\lambda| > r_K(a) \}$. Then every point $\lambda$ of $\Omega$ is an isolated point and $e_{\lambda} \in K$.

**PROOF.** We first show that if $\lambda \in \Omega$, then $\lambda$ is a boundary point of $\sigma(a)$. If this is not the case we can find a neighbourhood of $\lambda$ which consists entirely of spectral points and does not intersect $\sigma_K(a)$. Let 

$t_0 = \sup\{ t \geq 0 : (1 + c)t \lambda \in \sigma(a) \text{ for all } 0 \leq c \leq t \}$

Then $(1 + t_0)\lambda$ is in the boundary of $\sigma(a)$ and it is not isolated. On the other hand, since $(1 + t_0)|\lambda| > r_K(a)$, $(1 + t_0)\lambda - a$ is Fredholm relative to $K$. It follows directly from [BMSW] Theorem R.2.4 that $(1 + t_0)\lambda$ is isolated, which yields a contradiction. Hence $\lambda$ is a boundary point of $\sigma(a)$. Applying [BMSW] Theorem R.2.4 again, this time to $\lambda$ instead of $(1 + t_0)\lambda$, it follows that $\lambda$ is an isolated spectral point. That $e_{\lambda} \in K$ follows from [BMSW] Lemma R.2.3.

**LEMMA 2.2** If $I$ is any two-sided ideal in $A$, then $I$ and ideal $\overline{I}$ which is the norm closure of $I$ have the same sets of idempotents.

**PROOF.** Let $p$ be an idempotent in $\overline{I}$. Then $p\overline{I}p$ is a Banach algebra with identity $p$, algebraic structure inherited from $A$ and norm $\|\cdot\|_p$ defined by 

$\|pap\|_p = \sup\{ \|(pap)(pbp)\| : b \in \overline{I}, \|pbp\| = 1 \}$. 

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