A NOTE ON $C_p$ ESTIMATES FOR CERTAIN KERNELS

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For a certain class of operators defined by integral kernels, a necessary and sufficient condition is given for the belonging to the Schatten-von Neumann ideals $C_p$. The operators considered generalize the classical Hankel operators; the results thus extend Peller's characterization of the Hankel operators in a class $C_p$.

In [10] Peller obtained a precise criterion for the Hankel operators $H_\phi$ to belong to a given Schatten-von Neumann class namely, $H_\phi \in C_p$ (1 $\leq p < \infty$) if and only if the antianalytic part of $\phi$ is in a certain Besov space on the unit circle (Recall that $H_\phi : H^2 + L^2 \otimes H^2$ is given by the formula $H_\phi f = (I - P_+)\phi f$, where $P_+$ is the Riesz projection from $L^2$ onto $H^2$). Peller extended subsequently his result to other operator ideals ([11]; see also [12], [14]).

On the other hand, since the commutator (in $L^2(\mathbb{T})$) of the multiplication operator $M_\phi$ with the Riesz projection has the matrix representation

$$
\begin{bmatrix}
0 & H_\phi \\
H_\phi & 0
\end{bmatrix}
$$

Rochberg [13] has suggested, as higher dimensional generalizations of Peller's result, the estimation of commutators of multiplication operators with singular integral operators of Calderón-Zygmund type. In this case the frame of $\mathbb{R}^d$ seems more natural. Results for such commutators and, more recently, for iterated commutators have been obtained by Janson and Wolff [7] Janson and Peetre [5]. For a comprehensive survey of Hankel operators, Peller's results and subsequent generalizations, see [9].

The present note considers a further generalization, suggested by the Fourier transform of the previous case. We obtain results for integral operators defined by a kernel of type:

$$A(x,y)\hat{\phi}(x-y)$$

where the main condition imposed on $A$ is its invariance under the action of a fixed discrete multiplicative subgroup $G$ of $\mathbb{R}^*_+$. 

(2) \[ A(gx, gy) = A(x, y), \quad x, y \in \mathbb{R}^d, \quad g \in G. \]

It seems improbable to obtain a necessary and sufficient characterization, in terms of both \( A \) and \( \phi \), of operators with kernel (1) belonging to a Schatten-von Neumann class. Our results are of the following type (see theorems 1 and 2): under certain conditions preimposed on \( A \), the operator defined by the kernel (1) is in \( C_p \) if and only if \( \phi \in \mathcal{H}^{d/p} \) (for the definition of the homogeneous Besov spaces used here, see [8]). This goal is actually achieved only for \( 1 \leq p \leq 2 \). For \( 2 < p < \infty \), we obtain only the necessity; however, the proof is more general and simpler than that in [7] or [10]. Note that in [5], about which we recently learned, kernels of type (1), but satisfying the stronger symmetry condition

\[ A(\lambda x, \mu y) = \lambda \mu A(x, y), \quad x, y \in \mathbb{R}^d, \quad \lambda, \mu > 0 \]

are mentioned as a possible further generalization of iterated commutators. Also, the use of interpolation in theorem 1 below is similar to that in [5].

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1. Let \( T(A, \phi) \) be the operator whose kernel is given by (1), and \( T(A) \) the operator with kernel \( A(x, y) \). We suppose that \( A \) is a locally integrable function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying condition (2) and that \( \phi \) belongs, say, to \( S(\mathbb{R}^d) \). We shall consider \( T(A) \) and \( T(A, \phi) \) as densely defined operators on \( L^2(\mathbb{R}^d) \). Suppose also that \( G \) is generated by \( g_0 > 1 \).

The following two lemmas provide the basic estimates. The proof of the first follows a technique of Peller [10], while the second is a straightforward computation.

**Lemma 1.** Consider \( E = \{(x, y) \mid 3\sqrt{d} \leq |x - y| \leq 3\sqrt{d^2}g_0^2\} \), and let \( \chi(x, y) \) be the characteristic function of some set \( E \supset E \). Then

\[ ||T(A, \phi)||_{C_1} \leq C a_1(A)||\phi||_{\mathcal{B}_1}^d \]

where \( a_1(A) = ||T(A)||_{C_1} \) (as everywhere below, \( C \) denotes a universal constant, not necessarily the same in different inequalities).