EXTENSIONS OF INTERTWINING RELATIONS

Vlastimil Pták and Pavla Vrbová

Let $U_1$ and $U_2$ be two isometries acting respectively on the Hilbert spaces $H_1$ and $H_2$, and let $M_1$ be a $U_1$-invariant subspace of $H_1$. Let $X: M_1 \rightarrow H_2$ be a bounded linear operator intertwining $U_2$ and the restriction to $M_1$ of $U_1$,

$$X(U_1 | M_1) = U_2 X.$$

The authors give necessary and sufficient conditions for the existence of a bounded operator $Y: H_1 \rightarrow H_2$ which extends $X$ and satisfies $Y U_1 = U_2 Y$.

In the course of their study of generalized Hankel operators the authors investigated, not long ago, intertwining relations of the form

$$X T_1^* = T_2 X$$

where $T_1$ and $T_2$ are given contractions acting on Hilbert spaces $H_1$ and $H_2$ respectively. The problem considered [3] was to lift the above identity to a relation of the form

$$Y U_1^* = U_2 Y$$

the $U_1$ being the minimal isometric dilations of the corresponding $T_1$. It turns out that there is an essential difference between this lifting problem and the classical one of Sarason-Nagy-Foiaş. Whereas relations of the form $X T_1 = T_2 X$ may always be lifted to $Y U_1 = U_2 Y$, in the case considered above (where $T_1^*$ is to be dilated to a coisometry $U_1^*$) an additional condition has to be imposed for the lifting to be possible. This additional requirement assumes the form of a stronger boundedness condition to be satisfied by $X$. The authors [4] called it $R$-boundedness: indeed, it is easy to see that any operator $Y$ intertwining $U_1^*$ and $U_2$,
\( V^k \) satisfies the relation \( Y = P(R_2)Y = YP(R_1) \) where the \( P(R_i) \) are the orthogonal projections onto the subspaces \( R_i \) such that \( U_i|P(R_i) \) is the unitary part in the Wold decomposition of \( U_i \). As a compression of an operator \( Y \) satisfying \( Y = P(R_2)Y P(R_1) \) the operator \( X \) clearly satisfies

\[
(Xh_1, h_2) = (Yh_1, h_2) = \langle YP(R_1)h_1, P(R_2)h_2 \rangle.
\]

It follows that

\[
| (Xh_1, h_2) | \leq | Y | | P(R_1)h_1 | | P(R_2)h_2 |
\]

and this condition is stronger in general than ordinary boundedness. The referee of [3] called the authors' attention to the fact that an extension problem for shift operators considered by L. B. Page [2] also requires an additional continuity condition in order to possess a solution, while the corresponding problem for coisometries is always solvable. To be more precise: given a coisometry \( S^* \) on a Hilbert space \( H \), an \( S^* \)-invariant subspace \( M \subseteq H \) and an operator \( T_0 \in B(M) \) which commutes with the restriction \( S^*|M \) it is possible to show that there exists an extension \( T \) of \( T_0 \) to the whole of \( H \) which commutes with \( S^* \) and such that \( |T| \leq |T_0| \). If \( S^* \) is replaced by an isometry the analogous extension result is false in general. Indeed, Page considers a unilateral shift \( S \) on a Hilbert space \( H \), \( M \) an invariant subspace for \( S \) and \( T_0 \) a bounded operator on \( M \) commuting with \( S|M \). He then shows that operators \( T_0 \) for which an extension is possible satisfy an additional condition in the absence of which the result may fail: he exhibits an example of a two-dimensional shift and an operator \( T_0 \) such that no extension of \( T_0 \) will commute with \( S \). (For a shift of multiplicity one an extension is always possible.)

In the same paper Page formulated the conjecture that the necessary condition is also sufficient for the existence of an extension. The conjecture was subsequently proved in the particular case when \( S \) is a finite dimensional shift by C. F. Schubert [5]. The general case remained open.

In the present paper the authors consider the more general case of isometries and replace the commutation relation by a relation intertwining two different isometries. This has another further advantage: by separating the domain and the range spaces of the given operator a deeper insight is